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"Symplectic Tate homology and the vortex equations"

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Bott localization formula:

M closed manifold

$\circlearrowleft_{S^1}$ smooth S^1-action

Equivariant cohomology: $H^\bullet_{S^1}(M) = H^\bullet(M \times_{S^1} ES^1)$

module over $H^\bullet(\underset{\substack{\shortparallel \\ \mathbb{CP}^\infty}}{BS^1}) = \mathbb{Q}[\hbar]$ $\quad |\hbar| = 2$ (degree)

$\mathbb{Q}(\hbar) = \mathrm{Frac}\, \mathbb{Q}[\hbar]$

Tate homology: $\hat{H}^\bullet_{S^1}(M) = H^\bullet_{S^1}(M) \otimes_{\mathbb{Q}[\hbar]} \mathbb{Q}(\hbar)$

localization

$M^{S^1} := \mathrm{Fix}(S^1)$ fixed point set

$\circlearrowleft_{S^1}$ trivial S^1-action

THM (Bott): $\hat{H}^\bullet_{S^1}(M) = \hat{H}^\bullet_{S^1}(M^{S^1})$.

Consequence: $H^\bullet_{S^1}(M) \otimes_{\mathbb{Q}[\hbar]} \mathbb{Q}(\hbar) = H^\bullet(M^{S^1}) \otimes_{\mathbb{Q}} \mathbb{Q}(\hbar) = H^\bullet(M^{S^1}, \mathbb{Q}(\hbar))$

This THM of Bott fails in infinite dimensions.

E.g. N manifold, $\mathcal{L}_N = C^\infty(S^1, N)$ free loop space

Have $S^1 \circlearrowright \mathcal{L}_N$ using obvious action (induced by action on loop domain: $r_* v(t) = v(t+r)$

$t, r \in S^1 = \mathbb{R}/\mathbb{Z}$

THM (Goodwillie): $H^*_{S^1}(\mathcal{L}_N) \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$

depends only on $\pi_1(N)$.

Definitions of S^1-equivariant Tate homology:

- Jones – Petrack
- Greenlees – May
- Tene

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Vortices:

$\mathbb{C}$ $\circlearrowleft_{S^1}$ moment map: $\mu(z) = m(|z|^2 - 1)$

$\lambda = x\,dy$ $\quad \mu^{-1}(0) = S^1 \subset \mathbb{C}$

Rabinowitz action functional

$$\mathcal{A}^\mu : \mathcal{L}_\mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(v, \eta) \longmapsto -\int v^* \lambda - \eta \int_0^1 \mu(v)\, dt$$

L^2-gradient:

$$\nabla \mathcal{A}^\mu (v, \eta) = \begin{pmatrix} \partial_t v + \eta X_{\mu(v)} \\ -\int_0^1 \mu(v)\, dt \end{pmatrix}$$

$(v, \eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}) \times C^\infty(\mathbb{R}, \mathbb{R})$

Gradient flow equation:

$$\begin{cases} \partial_s v = i\,(\partial_t v + \eta\, X_{\mu(v)}) = 0 \\ \partial_s \eta - \int_0^1 \mu(v)\,dt = 0 \end{cases}$$

cf. vortex equations $\begin{cases} \bar\partial_{A,i} v = 0 \\ \underbrace{\partial_s \eta}_{F} - \underbrace{\int_0^1 \mu\, dt}_{\text{"}\mu\text{"}} = 0 \end{cases}$

can get "usual vortex equations" by using a different metric

—

$\rho = |v|^2$

$\Delta\rho(s,1) = \int_0^1 e^{2\rho(s,t)}\,dt - 1$

NB: S^1-displaceable, i.e. $\exists\, F \in C_c^\infty(\mathbb{C}\times S^1, \mathbb{R})$ s.t.

compactly supported

$\varphi_F^1(S^1) \cap S^1 = \emptyset$

Consequence: $HF(\mathcal{A}^H) = \{0\}$

Floer homology of $\mathcal{A}^\mu$

S^1-action on $\mathcal{L}_{\mathbb{C}} \times \mathbb{R}$
$\quad\quad\quad\quad\quad\quad \ni v$

$r_* v(t) = e^{2\pi i r}\, v(t) \qquad t \in S^1$

$r_*(v,\eta) = (r_* v, \eta)$

$S^1 \subset \mathbb{C}$ not equivariantly displaceable

$\operatorname{Crit} \mathcal{A}^\mu = \coprod_{k\in\mathbb{Z}} S^1_k \;,\quad S^1_k \cong S^1 \quad \text{via } \left(t \mapsto e^{2\pi i (r+kt)}, k\right)_{r\in S^1}$

Maslov index : $\mu(S^1_k) = 2k$

$\mathrm{Crit}\, \mathcal{A}^\mu / S^1 \cong \mathbb{Z}$

By degree reasons, $HF^{S^1}_*(\mathcal{A}^\mu) = \begin{cases} \mathbb{Q} & , * = 2k \quad , k \in \mathbb{Z} \\ \{0\} & , \text{else} \end{cases}$

$= \hat{H}^{S^1}_*(pt)$ $\quad$ S^1-equivariant Tate homology of a point

$\mathbb{Z}$-action on $\mathcal{L}_\mathbb{C} \times \mathbb{R}$:

$v \in \mathcal{L}_\mathbb{C} = C^\infty(S^1, \mathbb{C}) \quad , k \in \mathbb{Z}$

$k_* v(t) = e^{2\pi i k t} v(t)$

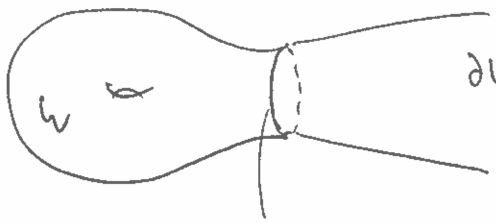
$k_*(v, \eta) = (k_* v, \eta + k)$

$d\mathcal{A}^\mu$ is $\mathbb{Z}$-invariant

<u>Symplectic Tate homology</u>:

(W, λ_W) Liouville domain

$d\lambda_W = \omega_W$ symplectic structure



$\partial W \times \mathbb{R}_+ \quad , \quad d(r \lambda_W|_{\partial W})$

$(\partial W, \lambda_W|_{\partial W})$ contact manifold

$H \in C^\infty(W, \mathbb{R})$ quadratic on $\partial W \times \mathbb{R}_+$

$\mathcal{A}_H : \mathcal{L}_W \longrightarrow \mathbb{R}$

$v \longmapsto -\int v^* \lambda_W - \int H(v)$

Symplectic homology is the Morse–Floer homology of this:

$$SH(W) = HF(\mathcal{A}_H)$$

$\mathcal{L}_W$

$\circlearrowleft S^1$ rotating domain

$$\mathcal{S} := \left(\mathcal{L}_W \times (\mathcal{L}_{\mathbb{C}} \times \mathbb{R}) \right) / S^1$$

↖ diagonal action

$$J_H\left([v_1, v_2; \eta] \right) = \mathcal{A}_H(v_1) + \mathcal{A}^\mu(v_2, \eta)$$

Symplectic Tate homology:

$$HT(W) := HF(J_H)$$

$\mathbb{Z}$-action on $\mathcal{L}_{\mathbb{C}} \times \mathbb{R}$ (commutes with S^1-action)

$\rightsquigarrow$ $\mathbb{Z}$-action on $\mathcal{S}$

Novikov field

$$\mathcal{N} \ni \xi = \sum_{i=-\infty}^{\infty} \underset{\mathbb{Q}}{\underset{\cap}{\xi_i}} t^i \qquad \exists\, i_0 = i_0(\xi) \text{ s.t. } \xi_i = 0 \ \forall i \geq i_0$$

$\Rightarrow HT(W)$ vector space over $\mathcal{N}$

Fixed point property $\Rightarrow HT(W) = H(W; \mathcal{N})$

But: have filtration on $HT(W)$ by $\mathcal{A}^\mu$.

$a \in \mathbb{R} \longrightarrow HT_a(W)$ generated by critical points c of J_H s.t. $\mathcal{A}^\mu(c) \geq a$.

Fact: $\varprojlim_{a\to\infty} HT_a(W) = \varprojlim_{\hbar} SH^{S^1}(W)$

S^1-equivariant symplectic homology as defined by Bourgeois – Oancea

$\varprojlim$, H do not commute in general.

Milnor's exact sequence

1st derived functor of $\varprojlim$

$$0 \longrightarrow \varprojlim{}^1 HT_a(W) \longrightarrow HT(W) \longrightarrow \varprojlim HT_a(W) \longrightarrow 0$$

EXAMPLE: (N, g) closed Riemann manifold, spin.

T^*N Liouville domain.

THM (Abbondandolo – Schwarz, Salamon – Weber, Viterbo, Kragh):

orientations

$$SH^{S^1}(T^*N) = H^{S^1}(\mathcal{L}_N)$$

Goodwillie: $\varprojlim HT_a(T^*N)$ only depends on $\pi_1(N)$

If N is simply connected,

$$\varprojlim HT_a(T^*N) = \varprojlim HT_a(T^*pt) = \mathcal{N}$$

But $HT(T^*N) = H(N; \mathcal{N})$.

((Ring structure: (cf. Chas - Sullivan) ?))