Abelian Vortices Revisited

J.M. Baptista

IST - Lisbon

Workshop GGV - HMI - Bonn - 27/11/2012

Plan of the talk

PART 1 - Degenerate metrics

- Abelian vortices interpreted as degenerate metrics on Riemann surfaces.
- Abelian vortices interpreted as degenerate metrics on Kähler manifolds.

PART 2 - Vortices on Kähler manifolds

- Vortices on a line bundle $L \to M$ over a Kähler manifold: L^2-metric, volume and Kähler class of the moduli space $\mathcal{M}$.
- Abelian GLSM on a Kähler manifold: moduli spaces and metrics.
- $\mathcal{M}$ as a compactification of the space $\mathcal{H}$ of holomorphic maps $M \to \mathbb{CP}^k$. Conjectures for the volume of $\mathcal{H}$.

Vortex Equations

- The simplest setting:
  - (M, ω_M) is a complex Kähler manifold;
  - $L \longrightarrow M$ is a complex line bundle with a hermitian metric.

- The fields of the theory are:
  - A $U(1)$-connection A on the line bundle L;
  - A section ϕ of L.

- The "self-dual" energy functional is

$$E(A, \phi) \;=\; \int_{(M,\omega_M)} \frac{1}{e^2}|F_A|^2 \;+\; |\mathrm{d}^A\phi|^2 \;+\; e^2\big(|\phi|^2 - \tau\big)^2$$

  where
  - $F_A \;=\;$ curvature of A;
  - $\mathrm{d}_A\phi \;=\;$ covariant derivative of ϕ;
  - $e^2, \tau \;=\;$ positive constants.

- In 2D, i.e. when M is a Riemann surface, the vortex equations are

$$\begin{aligned} \bar{\partial}_A \phi &= 0 \\ * F_A - i e^2 \left(|\phi|^2 - \tau \right) &= 0 \end{aligned}$$

where $*$ is the Hodge star operator on M.

- When M is a Kähler manifold, the vortex equations are

$$\begin{aligned} \bar{\partial}_A \phi &= 0 \\ \Lambda F_A - i e^2 \left(|\phi|^2 - \tau \right) &= 0 \\ F_A^{0,2} &= 0 \,, \end{aligned}$$

where ΛF_A is the contraction of the Kähler form ω_M with F_A.

- The "smooth part" of the moduli space $\mathcal{M}$ has a natural Kähler structure.

- If $(\dot{A}, \dot{\phi})$ is a tangent vector to the space of connections and sections, the L^2-metric on $\mathcal{M}$ is determined by the norm

$$\| \, (\dot{A}, \dot{\phi}) \, \|^2_{T\mathcal{M}} \;:=\; \int_{(M,\omega_M)} \frac{1}{e^2} \, |\dot{A}|^2 \;+\; |\dot{\phi}|^2$$

  applied to the component of $(\dot{A}, \dot{\phi})$ perpendicular to gauge transformations.

- The complex structure on $\mathcal{M}$ is determined by

$$J_{\mathcal{M}} \, (\dot{A}, \dot{\phi}) \;:=\; (J_M \, \dot{A}, \; i\dot{\phi})$$

  It is integrable, compatible with the metric on $\mathcal{M}$, and $(J_{\mathcal{M}})^2 = -1$.

An amusing interpretation of abelian vortices:

Abelian vortices in 2D

When M is a Riemann surface and $\deg L < e^2 \tau \operatorname{Vol} M$ the vortex moduli space is [Bradlow, García-Prada]

$$\mathcal{M} := \frac{\{\text{vortex solutions } (A, \phi)\}}{U(1)\text{-gauge transformations}}$$

$$\simeq \frac{\{\text{solutions of } \bar{\partial}_A \phi = 0 \text{ with non-zero } \phi\}}{\mathbb{C}^*\text{-gauge transformations}}$$

$$\simeq \Big\{\text{choices of unordered points } p_1, \ldots, p_d \in M, \text{ with } d = \deg L\Big\}$$

$$\simeq \text{ Symmetric product:} \quad \operatorname{Sym}^d M.$$

There are no vortex solutions with non-zero ϕ when $\deg L \geq e^2 \tau \operatorname{Vol} M$.

Vortices as degenerate metrics

- Let (M, ω) be a Riemann surface. Let ϕ be a holomorphic section of a line bundle $L \to M$. Given a hermitian metric h on L, define the degenerate Kähler form

$$\omega' \ := \ \tau^{-1} \, |\phi|_h^2 \, \omega \ .$$

- Consider the vortex equation

$$F_h \ - \ i \left(|\phi|_h^2 \ - \ \tau \right) \omega \ ,$$

where F_h is the curvature of the Chern connection on (L, h).

- The hermitian metric h satisfies the vortex equation if and only if

$$F_{\omega'} - i\tau \, \omega' \ = \ F_\omega - i\tau \, \omega \ ,$$

away from the zeroes of ϕ, where F_ω is the curvature of the Riemannian metric ω.

Vortices as degenerate metrics

- Studying abelian vortices on (M, ω) is the same thing as studying Kähler metrics ω' that satisfy the curvature equation.

- Given an effective divisor $D = \sum n_j \, q_j$ on M, there exists a unique Kähler metric ω' on M that satisfies the curvature equation and is degenerate at the points q_j with multiplicity n_j.

- Away from the degenerate points q_j, we have that

$$F_h = F_{\omega'} - F_{\omega} \ .$$

Since the curvature (or magnetic flux) of a vortex solution is concentrated around the zeros of the Higgs field ϕ, we see that $F_{\omega'}$ differs the most from F_{ω} around the degeneracy points of ω'.

Degenerate metrics

- Consider a general degenerate metric

$$\omega' = |z|^{2n_j} \, f_j \, \omega \ ,$$

where f_j is a smooth and positive function around the point $z = 0$.

- The Levi-Civita (or Chern) connection is given by:

$$A_{\omega'} \ = \ A_\omega \ + \ n_j \, z^{-1} \mathrm{d}z \ + \ \partial(\log f_j)$$

in a local holomorphic trivialization of TM. The connection is singular at $z = 0$. The residues n_j can be probed by integrating $A_{\omega'}$ around small circles centered at the singularity.

- The curvature is given by

$$F_{\omega'} \ = \ \mathrm{d}A_{\omega'} \ = \ F_\omega \ + \ \bar{\partial}\partial(\log f_j)$$

(plus a delta function at $z = 0$).

Degenerate metrics

- For a general degenerate metric on M, we have:

$$\frac{i}{2\pi} \int_{M \setminus \{q_j\}} F_{\omega'} \ = \ (2-2g) \, + \, \sum_j n_j \ ,$$

where g is the genus of M.

- Integrating the curvature equation

$$F_{\omega'} - i\tau\, \omega' \ = \ F_{\omega} - i\tau\, \omega$$

over $M \setminus \{q_j\}$ and using that $\int_M \omega' \geq 0$, we conclude that

$$2\pi \sum_j n_j \ \leq \ \tau\, (\mathrm{Vol}\, M) \ ,$$

which of course is the standard Bradlow condition for existence of vortex solutions.

Example: hyperbolic surfaces

- Suppose that the metric ω on the surface has constant scalar curvature $-\tau$. The equation for the degenerate metric reduces to

$$F_{\omega'} - i\tau\,\omega' \ = 0\ .$$

  Hence ω' also has curvature $-\tau$.

- Conclusion: on a hyperbolic surface (M, ω) all vortex solutions are given by quotients ω'/ω of hyperbolic metrics.

- For $M = \mathbb{H}$, all vortex solutions can be explicitly written as quotients $(f^*\omega)/\omega$, where the degenerate $\omega' = f^*\omega$ are pullbacks of the Poincaré metric by Blaschke products $f : \mathbb{H} \to \mathbb{H}$. [Witten 1977]

- On compact M, some (isolated) vortex solutions are given by quotients $(f^*\omega_Y)/\omega$, where (Y, ω_Y) is a second hyperbolic surface and $f : M \to Y$ is a ramified holomorphic map. [Manton-Rink 2010]

Vortices as degenerate metrics

- The curvature equation for ω' can be obtained as a minimizer of the functional

$$E(\omega') \;=\; \int_{(M,\omega)} \frac{1}{2}\,|F_{\omega'} - F_\omega|^2 \;+\; 2\tau\,|\mathrm{d}\sqrt{\omega'/\omega}\,|^2 \;+\; \frac{\tau^2}{2}\,|\omega' - \omega|^2 \;.$$

- If we allow vortex solutions on $L \to (M,\omega)$ with parabolic singularities, we obtain metrics ω' that degenerate as $|z|^{2\alpha_j}$ with $\alpha_j > 0$ not necessarily integer.

- If we allow meromorphic vortices on surfaces with singular Kähler metrics, the equation

$$F_{\omega'} - i\tau\,\omega' \;=\; F_\omega - i\tau\,\omega \;,$$

exposes two symmetries of the vortex equations that are less explicit in their usual form.

Vortices as degenerate metrics

Looking at $F_{\omega'} - i\tau\,\omega' \;=\; F_{\omega} - i\tau\,\omega$ we have:

- (1) if ω' is a solution on (M, ω), then ω is a solution on (M, ω').
- (2) if ω' is a solution on (M, ω) and ω'' is a solution on (M, ω'), then ω'' is a solution on (M, ω).
- (1) If (L, ϕ, h) is a vortex solution on (M, ω), then $(L^{-1}, \phi^{-1}, h^{-1})$ is a solution on $(M, \tau^{-1}|\phi|^2_h\,\omega)$.
- (2) If (L_1, ϕ_1, h_1) is a vortex solution on (M, ω) and (L_2, ϕ_2, h_2) is a solution on $(M, |\phi|^2_{h_1}\,\omega)$, then $(L_1 L_2, \phi_1\phi_2, \tau^{-1}\,h_1\,h_2)$ is a solution on (M, ω).

Vortices as degenerate metrics – Kähler manifolds

- Let $L \to (M, \omega)$ be a line bundle over a Kähler manifold. Given a holomorphic section ϕ and a hermitian metric h on L, consider the function $|\phi|_h^2$ on M.

- Let $V \to M$ be a vector bundle with a hermitian metric f. Define

$$f' := \tau^{-1} |\phi|_h^2 \, f \qquad\qquad \omega' := \tau^{-1} |\phi|_h^2 \, \omega \, .$$

- The metric h satisfies the vortex equation

$$\Lambda_\omega F_h - i \left(|\phi|_h^2 - \tau \right) = 0 \, ,$$

if and only if the rescaled (f', ω') satisfy

$$\left(\Lambda_{\omega'} F_{f'} - i\tau \right) \omega' = \left(\Lambda_\omega F_f - i\tau \right) \omega$$

away from the zeroes of ϕ.

Vortices as degenerate metrics – Kähler manifolds

- In particular, if the initial pair (f, ω) solves the Hermitian-Einstein equation $\Lambda_\omega F_f \ = \ i\tau$, so does the degenerate rescaled pair (f', ω').
- If the initial pair (f, ω) has constant scalar curvature $\mathrm{Tr}(\Lambda_\omega F_f) = i\tau \dim_{\mathbb{C}} X$, so does the degenerate rescaled pair (f', ω').
- Observe that if $\dim_{\mathbb{C}} X > 1$, the rescaled form $\omega' \ := \ \tau^{-1} \, |\phi|^2_h \, \omega$ is not d-closed on M, so defines only a Hermitian (non-Kähler) metric.

Modified vortex equation

- Let $L \to (M, \omega)$ be a line bundle over a Kähler manifold. Given a holomorphic section ϕ consider the equation for a hermitian metric h on L:

$$\Lambda_\omega F_h \; - \; \frac{i}{\dim_{\mathbb{C}} M} \Big(|\phi|_h^2 \; + \; s_\omega \Big) \; = \; 0 \; ,$$

where s_ω is the scalar curvature on (M, ω). It is

- When M is a Kähler surface, this equation is equivalent to a perturbation of the Seiberg-Witten equations.

- Call $\bar{s}_\omega < 0$ the average scalar curvature on (M, ω). Define

$$\omega' \; := \; -(\bar{s}_\omega)^{-1} \, |\phi|_h^2 \; \omega \; .$$

- The hermitian metric h satisfies the modified vortex equation if and only if the degenerate metric ω' has constant scalar curvature $s_{\omega'} \; = \; \bar{s}_\omega$.

Part 2: moduli spaces of vortices on Kähler manifolds

Abelian vortices on Kähler manifolds

If the parameter $\sigma := \left(\tau - \frac{\dim M}{e^2} \frac{c_1(L)^{\|}}{[\omega_M]}\right) \mathrm{Vol}\, M$ is positive, then

$$\mathcal{M} := \frac{\{\text{vortex solutions } (A, \phi)\}}{U(1)\text{-gauge transformations}}$$

$$\simeq \frac{\{\text{solutions of } \bar{\partial}_A \phi = 0 = F_A^{0,2} \text{ with non-zero } \phi\}}{\mathbb{C}^*\text{-gauge transformations}}$$

$$\simeq \frac{\{\text{pairs ``holomorphic structure on } L + \text{non-zero section } \phi\text{''}\}}{\mathbb{C}^*\text{-rescaling of the section}}$$

$$\simeq \left\{\text{effective divisors } D \text{ on } M \text{ with class } [D] \text{ Poincaré-dual to } c_1(L)\right\}$$

There are no vortex solutions with non-zero ϕ when $\sigma \leq 0$.

- An effective divisor $D \subset M$ is a choice of complex hypersurfaces in M. So when M is a Riemann surface, a divisor is just a choice of points in M.
- The divisor $D \subset M$ corresponding to a vortex solution (A, ϕ) is the hypersuface in M where the section ϕ vanishes.
- Equivalence of these different viewpoints for $\mathcal{M}$ was first proved by Bradlow using classic analytical results of Kazdan-Warner.
- Later, we will use a generalization of these results to GLSM, i.e. to vortices with gauge group $U(1)^k$ acting on a complex vector space with integer weights Q^a_j.

INFORMAL OBSERVATION:

- Given a divisor $D \subset M$ on a Kähler manifold, there exists a canonical 1-parameter family of curvature forms F_A^t on M such that:
  - The cohomology class $[F_A^t]$ is Poincaré-dual to $[D]$;
  - As $t \to 0$ the curvature $\Lambda F_A^t \to$ const. is Hermitian-Einstein.
  - As $t \to +\infty$ the curvature ΛF_A^t diverges (perhaps concentrates?) around the hypersurface $D \subset M$.
- This curvature F_A^t comes from the vortex solution corresponding to the divisor D and constants $e^2(t)$ and $\tau(t)$.

- Recall that for $\sigma > 0$ the vortex moduli space is

$$\mathcal{M} \simeq \frac{\{\text{pairs "holomorphic structure on } L + \text{non-zero section } \phi\text{"}\}}{\mathbb{C}^*\text{-rescaling of the section}}$$

- Now, the different holomorphic structures on L are parametrized by the Picard group $\mathrm{Pic}^0(M)$.

- For each fixed holomorphic structure on L, the space of holomorphic sections ϕ is the vector space $H^0(M; L)$.

- So the space of pairs "holomorphic structure on L plus section ϕ" is a (sort of) vector bundle over $\mathrm{Pic}^0(M)$ with fibre $H^0(M; L)$.

- Taking the quotient by the $\mathbb{C}^*$-rescaling on the fibres, we see that $\mathcal{M}$ is a projective bundle over $\mathrm{Pic}^0(M)$.

Examples – simply connected M

- When M is simply connected the Picard group $\mathrm{Pic}^0(M)$ is just a point. So:

$$\mathcal{M} \simeq \mathbb{P}(H^0(M;L)) \simeq \mathbb{CP}^{r-1} ,$$

  where $r = \dim H^0(M;L)$. Particular cases are:

- When $M = \mathbb{CP}^m$ and $L \to M$ has degree d, then $r = \frac{(m+d)!}{m!\,d!}$.

- When $M = \mathsf{Gr}(n,k)$ and $L \to M$ has degree d, then $r = \prod_{i=1}^{n-k} \prod_{j=n-k+1}^{n} \frac{d+j-i}{j-i}$.

- In this case the cohomology ring of $\mathcal{M}$ is very simple.

Examples – abelian varieties

- When M is an abelian variety (higher-dimensional complex torus) the Picard group $\mathrm{Pic}^0(M) = \hat{M}$ is the dual torus. So $\dim M = \dim \hat{M}$.

- Given a positive line bundle $L \longrightarrow M$, the Fourier-Mukai transform $\hat{L} \longrightarrow \hat{M}$ is a vector bundle with fibre $H^0(M; L)$. It is defined as

$$\hat{L} \;=\; (p_2)_* \left(\mathcal{P} \otimes p_1^* L \right) \;,$$

where $\mathcal{P} \longrightarrow M \times \hat{M}$ is the Poincaré line bundle, and p_1, p_2 are the natural projections from the product $M \times \hat{M}$ onto the factors.

- It follows from the Hitchin-Kobayashi correspondence and an observation of Brion that

$$\mathcal{M} \;\simeq\; \mathbb{P}(\hat{L} \to \hat{M}) \;.$$

Examples – abelian varieties

- Using the properties of projective bundles and the Fourier-Mukai transform, it is possible to compute the cohomology ring of the moduli space $\mathcal{M}$. It depends on the total Chern class of $\hat{L}$.

- Calling $r = \dim H^0(M; L)$, we can use Grothendieck-Riemann-Roch to obtain

$$c(\hat{L}) \;=\; \Big(\frac{\exp\big[\arctan\big(c_1(\hat{L})/r\big)\big]}{\sqrt{1 + \big(c_1(\hat{L})/r\big)^2}} \Big)^r ,$$

  where one should equate the terms of the Taylor expansion order by order.

After the cohomology, lets look at the metric on $\mathcal{M}$:

Volume and Kähler class of the moduli space

- Recall the stability parameter $\sigma := \left(\tau - \frac{\dim M}{e^2} \frac{c_1(L)^{\|}}{[\omega_M]}\right) \mathrm{Vol}\, M > 0$.

- When M is simply connected we have $\mathcal{M} \simeq \mathbb{CP}^{r-1}$. Then the Kähler class of the L^2-metric on the moduli space is simply

$$[\omega_{\mathcal{M}}] = \sigma\, \eta$$

  where $\eta \in H^2(\mathbb{CP}^{r-1}, \mathbb{Z})$ is the positive generator of the cohomology.

- In particular the volume of the moduli space is

$$\mathrm{Vol}(\mathcal{M}, \omega_{\mathcal{M}}) = \frac{\sigma^{\,r-1}}{(r-1)!}$$

  and the total scalar curvature is

$$\int_{\mathcal{M}} s(\omega_{\mathcal{M}})\ \mathrm{vol}_{\mathcal{M}} = \frac{r\,(r-1)}{\left[(r-1)!\right]^{1/(r-1)}} \left(\mathrm{Vol}\,\mathcal{M}\right)^{(r-2)/(r-1)} .$$

Volume and Kähler class of the moduli space

- When M is an abelian variety of dimension m, we have that $\mathcal{M} \simeq \mathbb{P}(\hat{L} \to \hat{M})$ is a projective bundle over the dual torus $\hat{M}$.

- Then the Kähler class of the L^2-metric on the moduli space is

$$[\omega_{\mathcal{M}}] \; = \; \sigma\,\eta \; - \; \frac{1}{e^2}\,\mathcal{F}\Big(\frac{[\omega_M^{m-1}]}{(m-1)!}\Big)$$

  where the class η is the generator of the cohomology of projective fibres of $\mathcal{M}$, and $\mathcal{F} : H^{2m-2}(M;\mathbb{R}) \longrightarrow H^2(\hat{M};\mathbb{R})$ is the cohomological Fourier-Mukai transform.

- When M is an abelian variety of complex dimension 2,

$$\mathrm{Vol}(\mathcal{M},\omega_{\mathcal{M}}) \; = \; (\mathrm{Vol}\,M)\Big(\frac{\tau\,\sigma^r}{r!} \; + \; \frac{\sigma^{r-1}}{e^4\,(r-1)!}\Big) \; .$$

Volume and Kähler class of the moduli space

- These volumes and Kähler classes of $(\mathcal{M}, \omega_{\mathcal{M}})$ extend results of Manton & Nasir, Perutz, JB for the case $\dim_{\mathbb{C}} M = 1$.
- How does one compute these volumes of $\mathcal{M}$?
- The ingredients are:

  (1) the Chern classes of the universal bundle over $\mathcal{M} \times M$;
  (2) a generalization of a formula for $\omega_{\mathcal{M}}$ due to Perutz;
  (3) knowledge of the cohomology ring of $\mathcal{M}$;
- Time permitting, I will discuss this later...

And how can we generalize (complicate) this story?

Abelian GLSM

- All this can be generalized to abelian gauged linear sigma models, i.e. to the case where the gauge group is $U(1)^k$ and there are many matter fields ϕ_j that transform with integer weights Q_j^a.

- So A is a connection on a principal $U(1)^k$-bundle $P \longrightarrow M$. We consider the linear torus action on $\mathbb{C}^n$:

$$(g_1, \ldots, g_k) \cdot (z_1, \ldots, z_n) \longmapsto (\ldots, z_j \prod_{a=1}^{k} (g_a)^{Q_j^a}, \ldots)$$

Each component of this action defines an associated line bundle $L_j := P \times_j \mathbb{C} \longrightarrow M$. Then $\phi = (\phi_1, \ldots, \phi_n)$ is a section of the direct sum of line bundles $L_1 \oplus \cdots \oplus L_n \longrightarrow M$.

Abelian GLSM

- The vortex equations are now:

$$\begin{aligned} \bar{\partial}_A \phi &= 0 \\ \Lambda F_A - ie^2 \Big[\Big(\sum_j Q_j \, |\phi^j|^2 \Big) - \tau \Big] &= 0 \\ F_A^{0,2} &= 0 \, , \end{aligned}$$

  where τ is a constant in $i\mathbb{R}^k \simeq$ Lie $U(1)^k$.

- In the case of GLSM the vortex moduli space is:
  - For M simply connected, $\mathcal{M}$ is a toric manifold or orbifold.
  - For M an abelian variety, $\mathcal{M}$ is a toric fibration over a cartesian product $\times_k \hat{M}$.

- These toric manifolds are determined explicitly in terms of the weights Q_j^a and the stability parameter σ. Generalizes work of J. Wehrheim in 2D.

Abelian GLSM - Simply connected M

When the base M is simply connected, the vortex moduli space $\mathcal{M}$ is a toric orbifold. It can be simply described as:

- Recall that ϕ is a section of $L_1 \oplus \cdots \oplus L_n \longrightarrow M$.

- Consider the space of holomorphic sections
$$V = \bigoplus_{j=1}^{n} H^0(M, L_j) \ .$$
It has a natural $U(1)^k$-action with weights $Q_j^1, \ldots, Q_j^k$ on each summand $H^0(M, L_j)$.

- Then the vortex moduli space is the symplectic quotient
$$\mathcal{M} = V_{ss} \,/\, (\mathbb{C}^*)^k = V \,//\, U(1)^k$$

Abelian GLSM - abelian varieties

- Again, ϕ is a section of $L_1 \oplus \cdots \oplus L_n \longrightarrow M$. For each j, consider the Fourier-Mukai transforms $\hat{L}_j \longrightarrow \hat{M}$.

- Consider also the multiplication in the Picard group:

$$m_{\rho_j} : \times^k \,\mathrm{Pic}^0 M \;\longrightarrow\; \mathrm{Pic}^0 M \qquad\qquad (g_1, \ldots, g_k) \;\longmapsto\; \prod_{a=1}^{k} (g_a)^{Q_j^a}$$

- Then one can define the vector bundle

$$V \;:=\; \bigoplus_{j=1}^{n} m^*_{\rho_j} \hat{L}_j \;\longrightarrow\; \times^k \,\mathrm{Pic}^0 M \;.$$

  It has a natural $U(1)^k$-action with weights $Q^1_j, \ldots, Q^k_j$ on the fibres of each summand $m^*_{\rho_j} \hat{L}_j$.

- The vortex moduli space is the symplectic quotient

$$\mathcal{M} \;=\; V_{ss} \,/\, (\mathbb{C}^*)^k \;=\; V \,//\, U(1)^k$$

  So there is a natural projection $\mathcal{M} \longrightarrow \times^k \,\mathrm{Pic}^0 M$ with toric fibres.

Abelian GLSM - Kähler class

- For GLSM it is also possible to compute explicitly the Kähler class $[\omega_{\mathcal{M}}]$ of the L^2-metric on the moduli space. Both when M is simply connected and when it is an abelian variety.

- As we saw, if M is simply connected, the vortex moduli space $\mathcal{M}$ is a toric manifold. Toric manifolds have natural cohomology 2-classes $\eta_1, \ldots, \eta_k$ that generate the ring $H^*(\mathcal{M}, \mathbb{Z})$ (e.g. projective space has only one class η).

- Then the Kähler class of the L^2-metric is

$$[\omega_{\mathcal{M}}] = \sum_{a=1}^{k} \sigma^a \, \eta_a \qquad \in H^2(\mathcal{M}, \mathbb{Z}) ,$$

  where the coefficient is the stability parameter

$$\sigma := \left(\tau - \frac{\dim M}{e^2} \, \frac{c_1(P)^{\parallel}}{[\omega_M]} \right) \mathrm{Vol}\, M \qquad \in \mathbb{R}^k .$$

Holomorphic maps as vortices

- Let $L \longrightarrow M$ be a line bundle. Consider the vortex equations on $\oplus^n L$. So $\phi = (\phi_1, \ldots, \phi_n)$ is a multiple with n sections of L.

- A vortex solution $(A, \phi_1, \ldots, \phi_n)$ determines a holomorphic map $f_\phi : M \longrightarrow \mathbb{CP}^{n-1}$ by the formula

  $$f_\phi(x) \ = \ \left[\, \phi_1(x), \ldots, \phi_n(x) \,\right]$$

  This map is well defined iff:
  $(C1)$ the ϕ_j's do not vanish simultaneously at any $x \in M$.

- All holomorphic maps $M \to \mathbb{CP}^{n-1}$ are obtained in this way.

- The space $\mathcal{H}$ of holomorphic maps embedds as the subset $\mathcal{H} \hookrightarrow \mathcal{M}$ of vortex solutions that satisfy $(C1)$.

Relation between $\omega_{\mathcal{H}}$ and $\omega_{\mathcal{M}}$

- Take the natural metric $\omega_{\mathcal{H}}$ on the space of maps $\mathcal{H}$ defined by

$$\|\mathrm{d}f\|^2_{\omega_{\mathcal{H}}} \; = \; \int_{(M, g_M)} |\mathrm{d}f|^2 \; = \; \int_{(M, g_M)} (g_M)^{\alpha\beta} \, (\partial_\alpha f^j) \, (\partial_\beta f^l) \, (g_{\mathbb{CP}^{n-1}})_{lj}$$

  How does it relate to the vortex metric $\omega_{\mathcal{M}}$ through the embedding $\mathcal{H} \hookrightarrow \mathcal{M}$?

- We expect that as $e^2 \to \infty$ the vortex metric $\omega_{\mathcal{M}}(e^2) \; \to \; \omega_{\mathcal{H}}$ over the subset $\mathcal{H} \subset \mathcal{M}$.

- If $\mathcal{H} \hookrightarrow \mathcal{M}$ embeds as an open dense subset, we expect that

$$\mathrm{Vol}\,(\mathcal{H}, \omega_{\mathcal{H}}) \; = \; \lim_{e^2 \to \infty} \mathrm{Vol}\,(\mathcal{M}, \omega_{\mathcal{M}}) \; .$$

  So even though $\mathcal{H}$ is not compact and $\omega_{\mathcal{H}}$ cannot be extended to a smooth metric on $\mathcal{M}$, the volume $\mathrm{Vol}\,(\mathcal{H}, \omega_{\mathcal{H}})$ should be finite!

Example - conjecture

- Let $M = \mathbb{CP}^m$ be a projective space. Let the target be $X = \mathbb{CP}^{n-1}$ with the metric $\tau\, \omega_{\text{norm.FS}}$.

- Let $\mathcal{H}^d$ be the space of holomorphic maps $\mathbb{CP}^m \to \mathbb{CP}^{n-1}$ of degree $d > 0$.

- For $n > m$ the generic vortex solution satisfies $(C1)$, so $\mathcal{H}^d \hookrightarrow \mathcal{M}^d$ embeds as an open dense subset.

- The conjecture says that

$$\mathrm{Vol}\,(\mathcal{H}, \omega_{\mathcal{H}}) \;=\; \frac{\big[\,\tau\,(\mathrm{Vol}\,M)\,\big]^{nr-1}}{(nr-1)!}$$

where $r = \dim H^0(M, L) = (m+d)!/(m!\,d!)$.

- This formula was rigorously checked by Speight in the special case of maps $M = \mathbb{CP}^1 \longrightarrow \mathbb{CP}^{n-1}$ of degree 1.

- Similar conjectures can be made when M is an abelian variety or an arbitrary Riemann surface. We can also take $\mathcal{H}$ as the space of maps $M \longrightarrow X$ to a toric target. The formulae are more complicated.

Thank you!