Report on Research in Groups

Borderline Problems with Singular Integrals

May 15 - June 15, 2015 and May 22 - June 19, 2016

Organizers: Stefanie Petermichl, Maria Carmen Reguera, Alexander Volberg, Brett D. Wick

Topics

The research conducted under this program was is interested in harmonic analysis questions that are concerned with sharp estimates for singular integral operators, with specific interest in questions arising in weighted theory.

Roughly speaking, singular integral operators are those with kernels $K(x, y)$ that possess some singularity along the diagonal. The prototypical example is the Hilbert transform

$$\mathsf{H}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$

where the kernel just fails to be integrable in $\mathbb{R}$. Singular integral operators play a fundamental role in questions in complex analysis, operator theory, partial differential equations, function theory, and geometric measure theory. A deeper understand of sharp estimates related to singular integral operators is of fundamental importance and will find numerous applications in other areas of analysis. In the last 15 years this area has become much more active. Very difficult problems that had been set aside for many years or thought to be unapproachable were solved one after another.

Goals

The goals of this research program were to obtain weighted estimates for Calderón-Zygmund operators, and certain dyadic models of these operators.

Organization

The event was organized by Stefanie Petermichl, Maria Carmen Reguera, Alexander Volberg and Brett D. Wick.

Results

We obtained several results during this time. These include results about matrix weights and about non-homogeneous Calderón-Zygmund operators.

In terms of matrix weights, we obtained several different results. Participant Wick, in collaboration with Treil, Culiuc and Bickel, studied matrix weights and were able to give necessary and sufficient conditions for weighted L^2 estimates with matrix-valued measures of well localized operators. Namely, we seek estimates of the form:

$$\|T(\mathbf{W}f)\|_{L^2(\mathbf{V})} \leq C\|f\|_{L^2(\mathbf{W})}$$

where T is formally an integral operator with additional structure, $\mathbf{W}, \mathbf{V}$ are matrix measures, and the underlying measure space possesses a filtration. The characterization we obtain is of Sawyer-type; in particular we show that certain natural testing conditions obtained by studying the operator and its adjoint on indicator functions suffice to determine boundedness. Working in both the matrix weighted setting and the setting of measure spaces with arbitrary filtrations requires novel modifications of a T1 proof strategy; a particular benefit of this level of generality is that we obtain polynomial estimates on the complexity of certain Haar shift operators.

Participant Volberg, Petermichl, in collaboration with Nazarov and Treil, introduced the so called *convex body valued* sparse operators, which generalize the notion of sparse operators to the case of spaces of vector valued functions. They then proved that Calderón-Zygmund operators as well as Haar shifts and paraproducts can be dominated by such operators. By estimating sparse operators we obtain weighted estimates with matrix weights. We get two weight A_2-A_∞ estimates, that in the one weight case give us the estimate

$$\|T|_{L^2(\mathbf{W})\to L^2(\mathbf{W})} \leq C[\mathbf{W}]_{A_2}^{1/2}[\mathbf{W}]_{A_\infty} \leq C[\mathbf{W}]_{A_2}^{3/2}$$

where T is either a Calderón-Zygmund operator (with modulus of continuity satisfying the Dini condition), or a Haar shift or a paraproduct.

Participants Petermichl and Reguera formulated a bilinear version of an embedding theorem with matrix weights. In this version, the Carleson sequence itself is scalar and non-negative. The approach used resembles level set considerations using so-called outer measures. It was shown that TFAE.

$$\frac{1}{|K|}\sum_{Q\in D(K)} \alpha_Q \leq 1 \forall K \in D(Q_0)$$

$$\sum_{Q\in D(Q_0} \alpha_Q \left|\langle \langle \mathbf{W}\rangle_Q^{-1} \langle \mathbf{W}^{1/2} f\rangle_Q, \langle \mathbf{W}^{-1}\rangle_Q^{-1} \langle \mathbf{W}^{-1/2} g\rangle_Q\rangle_{\mathbb{C}^d}\right|$$

Work by Volberg, in collaboration with Zorin-Kranich, extended Lerner's recent approach to sparse domination of Calderón-Zygmnd operators to upper doubling (but not necessarily doubling), geometrically doubling metric measure spaces. Their domination theorem is different from the one obtained recently by Conde-Alonso and Parcet and yields a weighted estimate with the sharp power $\max(1, 1/(p-1))$ of the A_p characteristic of the weight.

Work by Domelevo, Ivanisvili, Petermichl, Treil, Volberg disprove lower square function estimates in the case of general (non-homgenous) discrete time filtrations. Concerned are inequalities of the form

$$\|f\|_{L^2(w)} \leq F(w)\|Sf\|_{L^2(w)}$$

where $F(w)$ can mean dependence of A_2 or A_∞ or mixed conditions. The authors show that the expected estimates known from the homogenous setting fail, in particular, the A_∞ condition is not sufficient for any lower estimate and the power of the estimate with an A_2 weight doubles when passing from the homogenous setting to general measures.