Representations of the affine (annular) BMW category

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Joint with Kevin Walker

. . . via a topological construction (from TQFTs)

representations

What is a representation of a category $\mathcal{B}$?

For each object $[n]$ of $\mathcal{B}$ assign a vector space $V_{[n]}$.
For each morphism $[m] \to [n]$ a linear map $V_{[m]} \leftarrow V_{[n]}$ compatible with other morphisms.

Example

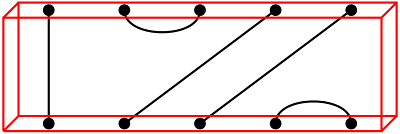
Recover representation of a group G, $G \to GL(V)$
Category has ONE object, so we just need one vector space V
Morphisms correspond to $g \in G$ with $g \circ h = gh$

$$V \xleftarrow{g} V \xleftarrow{h} V$$

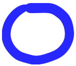
BMW category

Our finite BMW category $\mathcal{B}^{\text{fin}}$ has objects $[n]$ n marked (framed) points in a disk (or rectangle)

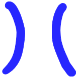
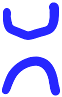
Morphisms described by these diagrams, subject to relations



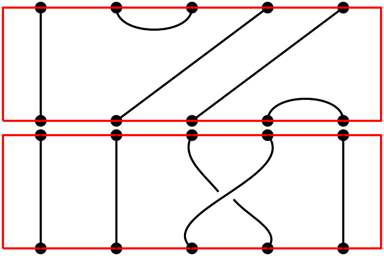
relations

 $= \delta = \frac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1$

 $= \alpha$ 

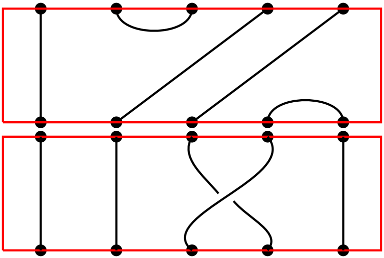
 $-$  $= (s - s^{-1})\Big($  $-$  $\Big)$

morphisms



Recover the BMW algebra B_n^{fin} as End$[n]$

morphisms



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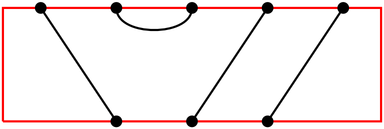
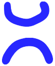


Example: $n = 2$ idempotents

$\frac{1}{\delta}E_1 =$ $\frac{1}{\delta}$ 

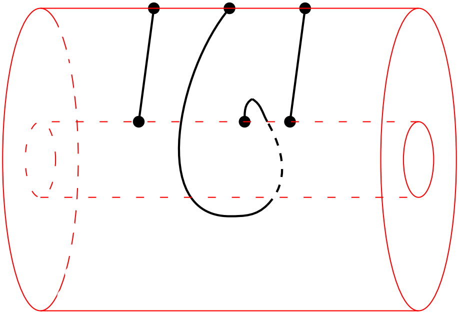
$\frac{1}{s+s^{-1}}\left(s^{-1} + T_1 + \frac{-\alpha^{-1}-s^{-1}}{\delta}E_1\right) = \frac{1}{s+s^{-1}}\Big(s^{-1}$  $+$  $+ \frac{-\alpha^{-1}-s^{-1}}{\delta}$  $\Big)$

$\frac{1}{s+s^{-1}}\left(s - T_1 + \frac{\alpha - s}{\delta}E_1\right) = \frac{1}{s+s^{-1}}\Big(s$  $-$  $+ \frac{\alpha-s}{\delta}$  $\Big)$

annular BMW category

Our affine BMW category $\mathcal{B}^{\mathrm{aff}}$ has object $[n]$ of n marked (framed) points in an annulus

We get "new" morphisms



representations

What is $V_{[n]}$?

We give it topologically. Take some manifold with a piece of its boundary that looks like a rectangle (annulus) with n marked points. Let tangles run around the manifold but begin/end at our marked points, subject to relations. This forms a vector space.

Morphisms act by thickening that piece of boundary up by a collar and gluing on our morphism.

Topology, TQFTs

This construction actually comes from topology (3-manifolds, $(3+1)$-dim TQFTs (actually $(3+\epsilon)$-dim TQFT), ...)

TQFT = topological quantum field theory

a TQFT is a machine for turning topology into algebra.

The BMW TQFT

- assigns to a 3-manifold M a vector space $V(M)$ (the BMW skein module). We actually get a family of vector spaces $V(M; c)$ depending on boundary conditions c on M.
- assigns a linear category $\mathcal{B}(Y)$ to a surface Y.
- If Y is contained in the boundary M, then the various vector spaces $V(M; c)$ afford a representation of $\mathcal{B}(Y)$.
- If a pair of surfaces $Y_1 \cup Y_2$ is contained in the boundary of M, then the various $V(M; c)$ constitute a $(\mathcal{B}(Y_1), \mathcal{B}(Y_2))$-bimodule.
- Gluing 3-manifolds along a surface Y corresponds to taking tensor product over $\mathcal{B}(Y)$

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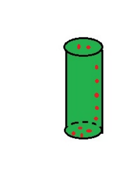
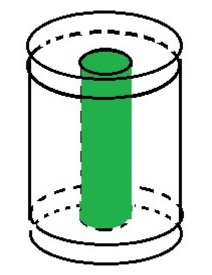
1-handle construction

bi-module $V(M)$ for $M = D^2 \times [0,1]$

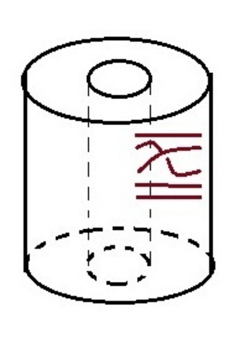


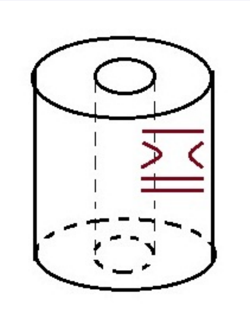
The boundary of the *solid* cylinder is TWO disks and an annulus.
When we draw the thickened boundary, this is where our morphisms from $\mathcal{B}^{\mathrm{fin}} \times \mathcal{B}^{\mathrm{fin}}$ and $\mathcal{B}^{\mathrm{aff}}$ can act

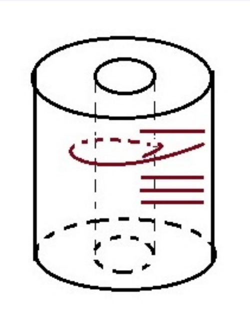


$\mathcal{B}^{\rm aff}$ corresponds to $Y = S^1 \times [0,1]$ annulus

Or we turn the picture sideways and draw (say T_2 and E_2 and $X_2 \in B_5^{\rm aff}$) as



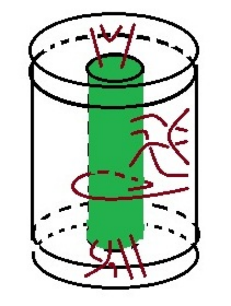




1-handle construction

bi-module

The $(\text{Rep}(\mathcal{B}^{\text{fin}}) \times \text{Rep}(\mathcal{B}^{\text{fin}}), \text{Rep}(\mathcal{B}^{\text{aff}}))$-bimodule structure:



We construct $M(\mu, \lambda, n)$ by imposing the finite irreps μ, λ (or idempotents) on the top/bottom of the cylinder/hatbox, and then let $\mathcal{B}^{\text{aff}}$ act on the remaining annular boundary.

We can apply this "machine" to other 3-manifolds ...

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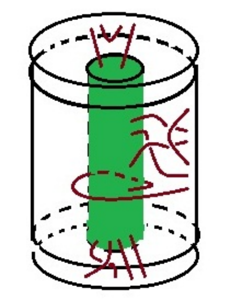

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Algebra and combinatorics

What is going on? As a representation of B_n^{aff}, what have we constructed? What is this functor

$$\mathrm{Rep}(\mathcal{B}^{\mathrm{fin}}) \times \mathrm{Rep}(\mathcal{B}^{\mathrm{fin}}) \to \mathrm{Rep}(\mathcal{B}^{\mathrm{aff}}).$$

What is this construction?

Start with two irreps: $M(\emptyset, \mu, m)$ of B_m^{fin}, $M(\emptyset, \lambda, \ell)$ of B_ℓ^{fin} and produce an X-ssl irrep $M(\mu, \lambda, n)$ of B_n^{aff}.

$$\mathrm{fin} \times \mathrm{fin} \to \mathrm{aff}$$

Definition

M is X-ssl if its restriction to $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is semisimple, i.e., if

$$M = \bigoplus_{\beta \in (\mathbb{C}^n)^*} M[\beta]$$

weight spaces

Let $\beta \in (\mathbb{C}^n)^*$. The β-weight space of M is

$$M[\beta] = \{v \in M \mid X_i v = \beta(X_i)v, 1 \leq i \leq n\}.$$

Fact

M is simple and X-ssl $\implies$ $\dim M[\beta] = 1$ or 0.
Determines a weight basis.
For simple M the $\beta \in \mathrm{sppt}\, M$ $(M[\beta] \neq 0)$ determine M.

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strategy

Fact: $M(\mu, \lambda, n)$ is an X-ssl irrep of B_n^{aff}.

Strategy: Construct a weight basis of $M(\mu, \lambda, n)$ topologically, and analyze its support.

The support matches another combinatorial construction of a B_n^{aff}-module (which we'll also call $M(\mu, \lambda, n)$). This will identify it. In other words, these modules must be isomorphic since they have the same support, i.e. their restrictions to X agree.

Side note: The construction of this "spine basis" has a Schur-Weyl flavor.

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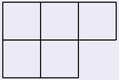
enter combinatorics— of known weight bases

Recall a partition λ with $|\lambda| = n$ indexes an irreducible representation of $\mathcal{S}_n$.

Example

$\lambda =$  indexes an irrep of $\mathcal{S}_5$.

Hecke algebra

or λ indexes an irrep $M(\emptyset, \lambda, 5)$ of the finite Hecke algebra H_5^{fin} of type A.

H_n^{fin} is a q-deformation of $\mathbb{C}[\mathcal{S}_n]$ with generators
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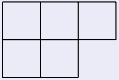
$\lambda =$  indexes an irrep of $\mathcal{S}_5$.

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basis of the module $M(\emptyset, \lambda, n) \longleftrightarrow \mathrm{SYT}(\lambda)$

1	2	3
4	5	

1	2	4
3	5	

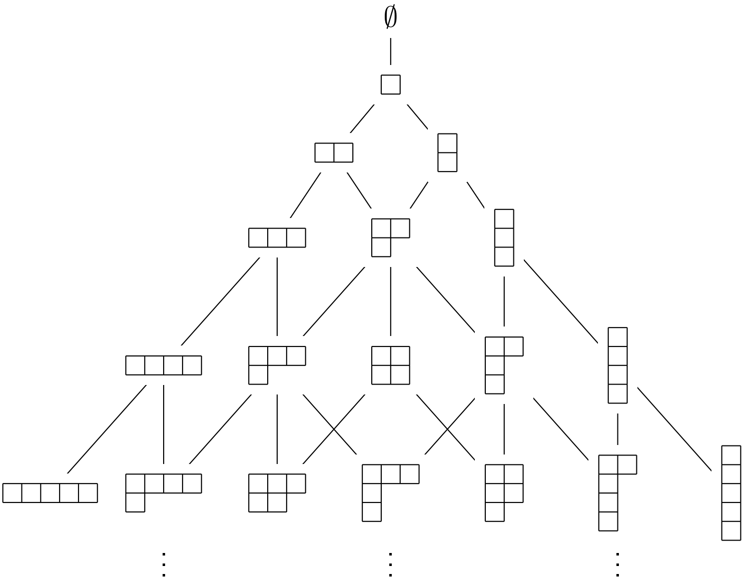
1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

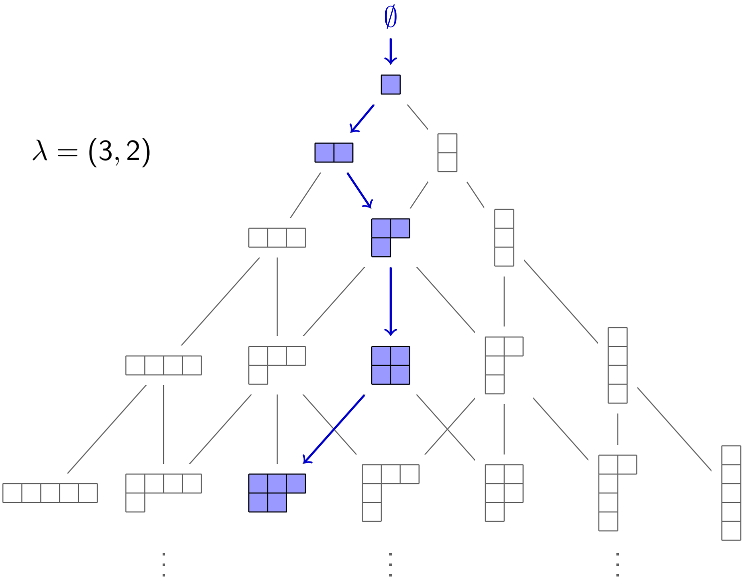
A standard Young tableau of shape λ ($\mathcal{T} \in \mathrm{SYT}(\lambda)$) corresponds to a path of length $n = |\lambda|$ from $\emptyset$ to λ in Young's lattice of partitions.

Young's lattice of partitions (is a crystal graph)

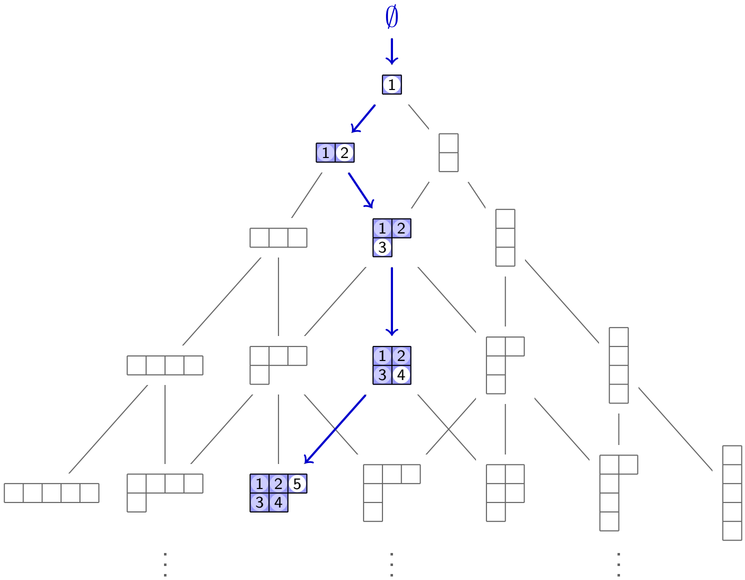


a path $\emptyset$ to λ

$\lambda = (3, 2)$



a path $\emptyset$ to $\lambda \longleftrightarrow \mathcal{T} \in STY(\lambda)$



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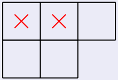
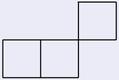
Why does this index a basis?   Induction/Restriction (among other reasons)

Skew shapes

Example

$\color{red}{\mu = (2)} \subseteq \lambda = (3, 2)$



$\lambda/\mu =$ 

Example (SYT)

$\mathrm{SYT}(\lambda/\mu)$

$\mathcal{T} \in \mathrm{SYT}(\lambda/\mu)$ corresponds to a path of length $n = |\lambda/\mu|$ from μ to λ in Young's lattice of partitions.

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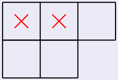

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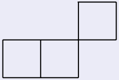

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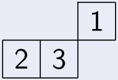
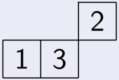
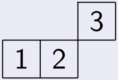
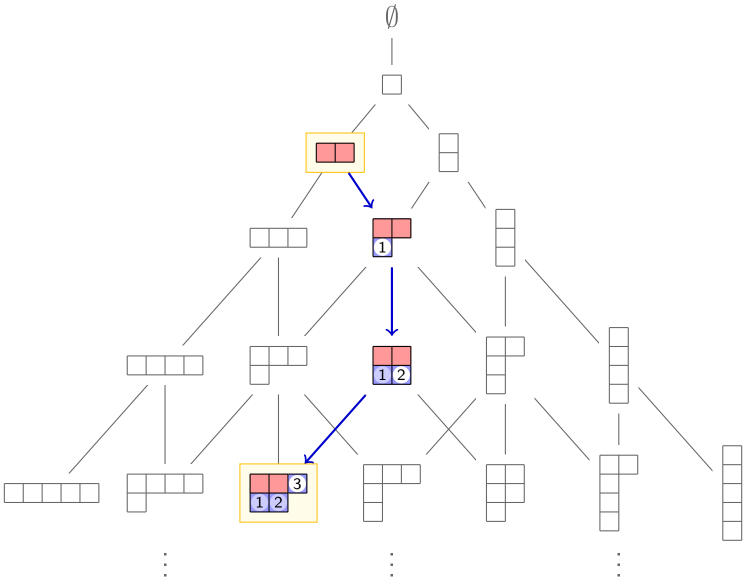






$\mathcal{T} \in \mathrm{SYT}(\lambda/\mu)$ corresponds to a path of length $n = |\lambda/\mu|$ from μ to λ in Young's lattice of partitions.

a path μ to λ



skew shapes

SYT(λ/μ) index a basis of an X-ssl irrep $M(\mu,\lambda,n)$ with $\mu \subseteq \lambda$,
$n = |\lambda| - |\mu|$.
of H_n^{aff}.

H_n^{aff} is the (extended) affine Hecke algebra of type A.

H_n^{aff} is a q-deformation of $\mathbb{C}[\mathcal{S}_n \ltimes \mathbb{Z}^n]$ with generators
T_i in place of $(i, i+1) = s_i \in \mathcal{S}_n$,
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As vector spaces, $H_n^{\mathrm{aff}} \simeq H_n^{\mathrm{fin}} \otimes \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

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finite vs affine

H_n^{fin}-modules have $\mu = \emptyset$, $n = |\lambda|$.

H_n^{fin} is a quotient of H_n^{aff} via

$$\begin{aligned} H_n^{\mathrm{aff}} &\twoheadrightarrow H_n^{\mathrm{fin}} \\ T_i &\mapsto T_i \\ X_1 &\mapsto 1 = q^0 \end{aligned}$$

For SYT, means $\boxed{1}$ can only be on 0^{th} diagonal

Fact

If we inflate any simple H_n^{fin}-module M via this homomorphism, it will be an X-ssl H_n^{aff}-module.
It will have a basis indexed by $\mathrm{SYT}(\lambda)$, i.e. will be $M(\emptyset, \lambda, n)$ for $n = |\lambda|$.

How to read the weights off the tableaux

X-ssl $M \in \mathrm{Rep}_q^{\mathrm{aff}}$, affine Hecke algebra case

$M[\beta] \ni v_{\mathcal{T}}$, $\mathcal{T} \in \mathsf{SYT}(\lambda/\mu)$

β_i describes which diagonal $\boxed{i}$ is on

The combinatorics of sppt $M = \{\beta \in \mathbb{Z}^n \mid M[\beta] \neq 0\}$ is that of $\mathsf{SYT}(\lambda/\mu)$, i.e. of n-step paths μ to λ.

Action of H_n^{aff} generators on basis $\{v_{\mathcal{T}} \mid \mathcal{T} \in \mathsf{SYT}(\lambda/\mu)\}$

$$X_i v_{\mathcal{T}} = q^{\mathrm{diagonal}\,\boxed{i}}\, v_{\mathcal{T}}$$

$$T_i v_{\mathcal{T}} \in span\{v_{\mathcal{T}}, v_{s_i\mathcal{T}}\}$$

If $s_i\mathcal{T} \notin \mathsf{SYT}(\lambda/\mu)$, set $v_{s_i\mathcal{T}} = 0$. (connection of SYT and weights; the algebra dictates the combinatorics)

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up-down

What if we now allow n-step paths to go up and down?

Then we capture the combinatorics of weights of X-ssl irreps of the affine BMW algebra B_n^{aff}. (See Leduc-Ram, Orellana-Ram)

n-step paths μ to λ $\longleftrightarrow$ basis of the B_n^{aff}-module $M(\mu, \lambda, n)$

Now μ, λ are arbitrary (we drop the requirement $\mu \subseteq \lambda$)
n is fixed, up to *parity*, independent of $|\mu|, |\lambda|$.

up-down

What if we now allow n-step paths to go up and down?

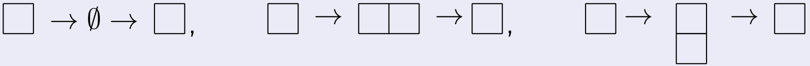
Then we capture the combinatorics of weights of X-ssl irreps of the affine BMW algebra B_n^{aff}. (See Leduc-Ram, Orellana-Ram)

n-step paths μ to $\lambda \longleftrightarrow$ basis of the B_n^{aff}-module $M(\mu, \lambda, n)$

Now μ, λ are arbitrary (we drop the requirement $\mu \subseteq \lambda$)
n is fixed, up to *parity*, independent of $|\mu|, |\lambda|$.

Example

$M((1),(1),2)$ has basis indexed by



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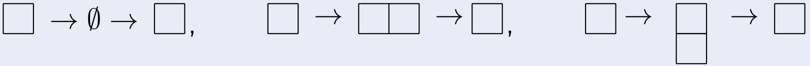

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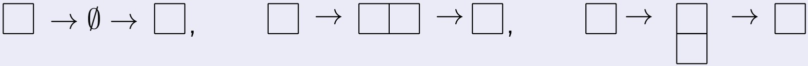

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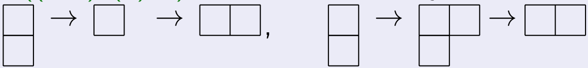

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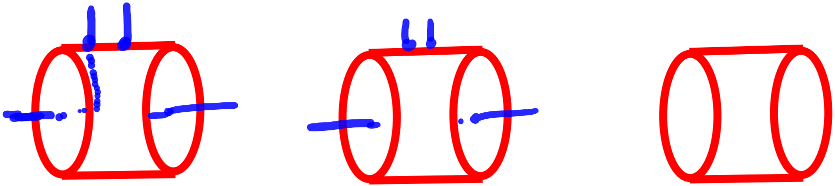
$\square \to \emptyset \to \square$, $\quad \square \to \square\square \to \square$, $\quad \square \to \begin{smallmatrix}\square\\ \square\end{smallmatrix} \to \square$



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When we now allow n-step paths to go up and down we capture the combinatorics of weights of X-ssl irreps of the affine BMW algebra B_n^{aff}. (See Leduc-Ram, Orellana-Ram)

B_n^{fin} is a deformation of the Brauer algebra, with generators T_i, E_i, $1 \leq i < n$.

E_{n-1} creates a link between B_n^{fin} and B_{n-2}^{fin}.

The irreps of B_n^{fin} correspond to λ with $n - |\lambda| \equiv 0 \mod 2$.

B_n^{aff} has generators T_i, E_i, X_i.

H_n^{aff} is a quotient of B_n^{aff} $(B_n^{\mathrm{aff}} \twoheadrightarrow H_n^{\mathrm{aff}})$ via

$$T_i \mapsto T_i$$
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Definition (Recall)

M is X-ssl if its restriction to $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is semisimple,
i.e., if

$$M = \bigoplus_{\beta \in (\mathbb{Z} \times \{+1,-1\})^n} M[\beta]$$

weight spaces

Let $\beta = (\beta_i, \varepsilon_i)_i \in (\mathbb{Z} \times \{+1, -1\})^n$. The β-weight space of M is

$$M[\beta] = \{v \in M \mid X_i v = s^{\beta_i} \alpha^{\varepsilon_i} v, 1 \leq i \leq n\}.$$

Fact

M is X-ssl $\implies$ dim $M[\beta] = 1$ or 0.
Determines a weight basis.

X-ssl $M \in \mathrm{Rep}_q^{\mathrm{aff}}$, BMW case

$M[\beta] \ni v_{\mathcal{T}}$, $\mathcal{T}$ is an n-step path λ to μ

β_i describes which diagonal $\boxed{i}$ is on

$\varepsilon_i \in \{+1, -1\}$ describes wether we add/remove the box

The combinatorics of sppt $M = \{(\beta, \varepsilon) \in \mathbb{Z}^n \times \{\pm 1\}^n \mid M[\beta] \neq 0\}$ is that of the n-step paths μ to λ.

Action of B_n^{aff} generators on basis $\{v_{\mathcal{T}}\}$

$$X_i v_{\mathcal{T}} = s^{\mathrm{diagonal}\,\boxed{i}} \alpha^{\varepsilon_i} v_{\mathcal{T}}$$

$$T_i v_{\mathcal{T}} \in span\{v_{\mathcal{T}'} \mid \mathcal{T}' \text{ differs from } \mathcal{T} \text{ only on the } i\text{th partition}\}$$

$$E_i v_{\mathcal{T}} = \begin{cases} 0 & \text{the } i-1\text{st and } i+1\text{st partition of } \mathcal{T} \text{ differ} \\ \in span\{v_{\mathcal{T}'} \mid \mathcal{T}' \text{ differs from } \mathcal{T} \text{ only on the } i\text{th partition}\} & else. \end{cases}$$

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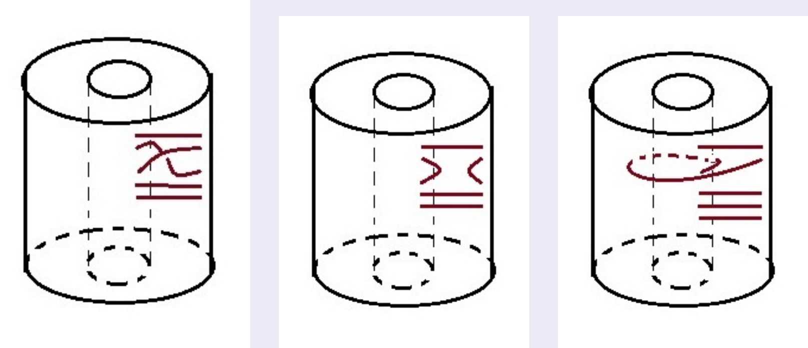
$M(\mu, \lambda, n)$

We can recover the action of all the generators of B_n^{aff} on $M(\mu, \lambda, n)$



Facts

- a directed n-step path on Young's lattice $\longleftrightarrow$ some $\mathcal{T} \in \mathrm{SYT}(\lambda/\mu)$ $\longleftrightarrow$ some weight β
- an n-step path on Young's lattice $\longleftrightarrow$ some weight $\beta = (\beta_i, \varepsilon_i)_{i=1}^n$
- These are *all* the allowable weights across all X-ssl modules in $\mathrm{Rep}_q^{\mathrm{aff}}$.
- Given X-ssl irrep $M \in \mathrm{Rep}_q^{\mathrm{aff}}$, the pair (μ, λ) is unique (up to diagonal shift in H_n^{aff} case).

$M(\mu, \lambda, n)$

Question

Are these **all** the irreps?

NO

Why

- Because boxes are in $\mathbb{Z}^2$ (not $\mathbb{C}^2$) we only get representations in $\text{Rep}_q^{\text{aff}}$, the full subcategory on which $\{X_i\}$ take eigenvalues in $\{\alpha^\ell s^k \mid \ell, k \in \mathbb{Z}\}$. (like integral weights)
- The configuration of boxes for skew shapes yield the X-semisimple (aka *calibrated*) representations. (see A. Ram)

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Other constructions, directions

other 3-manifolds M

other ways of slicing up the boundary (to yield a bi-module)

gluing and tensor product

non X-ssl representations?? (topology and category have rigidity)

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