2d TQFT’s and partial fractions

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Monoidal and 2-categories in representation theory and categorification

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(joint with M.Khovanov and Y.Kononov)

Definition (M.Atiyah): A TQFT is a symmetric tensor functor Cob_d→Vec.

Generalization: replace Vec by a symmetric tensor category C
C-valued TQFT: symmetric tensor functor Cob_d→C.

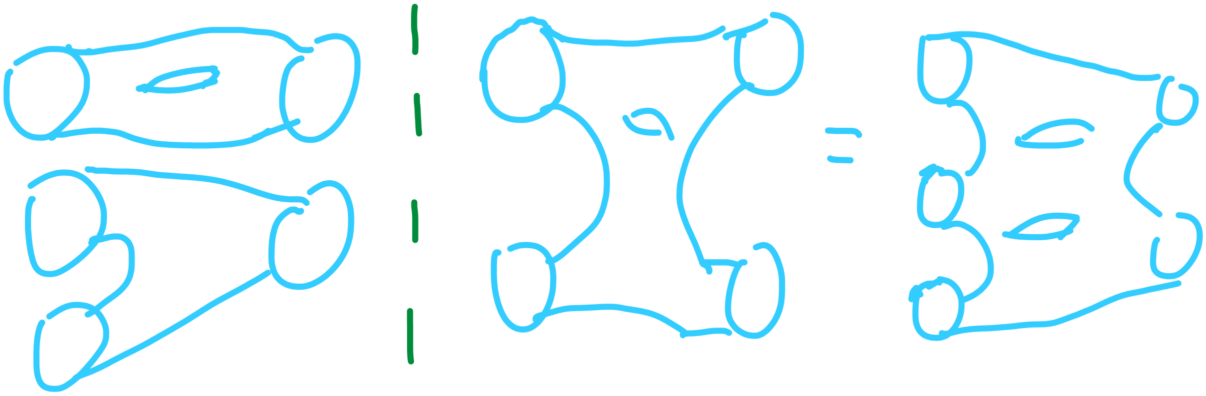
Examples: C=Vec, sVec, Rep(G) etc

Today: d=2!

Category of cobordisms Cob_d:

Objects: (d-1)-dimensional closed oriented manifolds

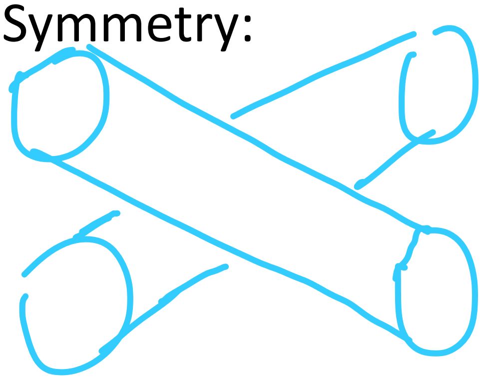
Morphisms: d-dimensional oriented cobordisms
Composition: gluing



Identity morphisms=?

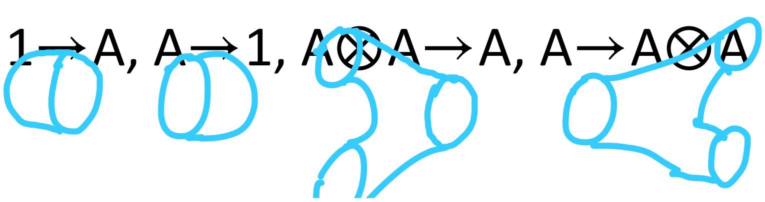
Tensor product: disjoint union
Unit object=?

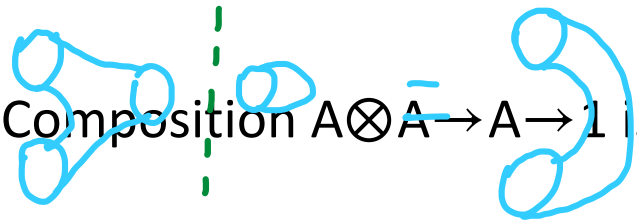
Symmetry:



Some important objects of Cob_2:
empty set=1 and circle=A

Some important morphisms:
1→A, A→1, A⊗A→A, A→A⊗A



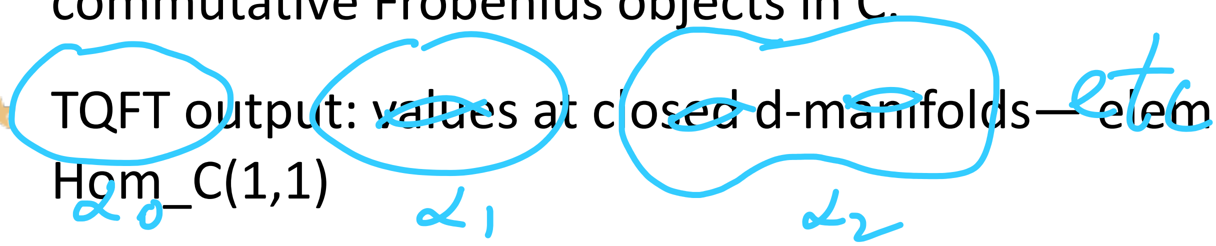


Composition $A\otimes A\to A\to 1$ is non-degenerate

- Thus A is a commutative Frobenius object in Cob_2: commutative associative unital monoid equipped with a map $A\to 1$ such that the composition $A\otimes A\to A\to 1$ is a non-degenerate pairing.

<u>Theorem</u> (R.Dijkgraaf + folklore): Cob_2 is free category generated by the commutative Frobenius object A.

<u>Corollary</u>: C-valued 2d TQFT's = Functors Cob_2$\to$C = commutative Frobenius objects in C.



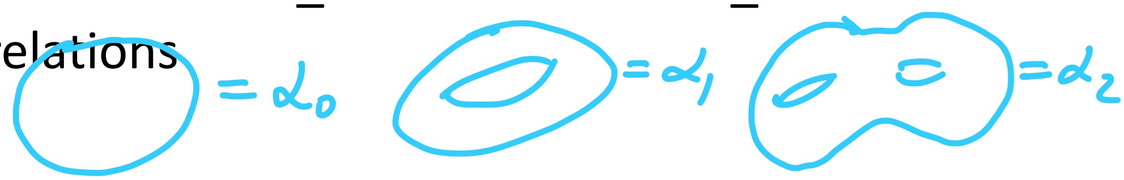
- TQFT output: values at closed d-manifolds — elements of Hom_C(1,1)

Linear setup: choose a field k
C — k-linear category
Hom_C(1,1)=k

Frobenius object = Frobenius algebra

TQFT output: sequence $\alpha_0,\ \alpha_1, \alpha_2,$ .... of elements of k
Answer 1: all sequences appear.

★ Main Question: Which sequences we will observe?
Take C = VCob_α:= linearized Cob_2 modulo relations



etc

★ Realization of sequence $\alpha = (\alpha_0,\ \alpha_1, \alpha_2,$ ....) : pair (C,A) such that the corresponding TQFT outputs α.

Realization is finite if Hom spaces in C are finite dimensional.

Answer 2 (M.Khovanov): sequence α admits a finite realization if and only if it is linearly recursive, that is the generating function Z(T)= $\alpha_0 +\ \alpha_1 T + \alpha_2 T^2 + \ldots$ is rational.

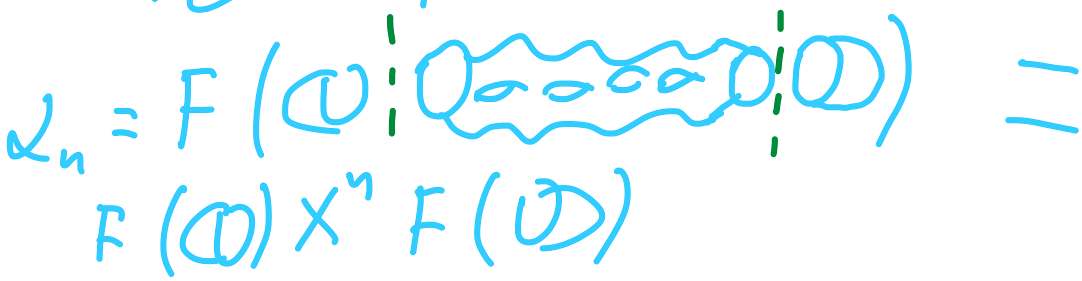
$x = F($  $)$

★ $1, x, x^2, x^3, \ldots$ are linearly dependent

e.g. $x^3 - 2x^4 + 3x^5 = 0$

hence $x^n - 2x^{n+1} + 3x^{n+2} = 0$ for $n \geq 3$

hence $\alpha_n - 2\alpha_{n+1} + 3\alpha_{n+2} = 0$ for $n \geq 3$



Existence: some quotients SCob_α of linearized VCob_2.

Realization is abelian if C is abelian, rigid, with finite dimensional Hom spaces.

<u>Answer 3</u> (M.Khovanov, Y.Kononov, V.O.): sequence α admits an abelian realization if and only if it is

1) linearly recursive, $Z(T)=\frac{P(T)}{Q(T)}$ with relatively prime P(T) and Q(T).

2) Q(T) has no multiple roots (in K) and deg P $\leq$ deg Q+1.

Thus $Z(T)=\delta_0 + \delta_1 T + \sum_i \frac{\beta_i}{1-\gamma_i T}$

3) If char k=p>0 then δ_1 and $\beta_i\gamma_i$ (for all i) are in the prime

subfield $F_p \subset$ k

Remark: conditions 2) and 3) can be expressed in terms of 1-form Z(T)$\frac{dT}{T^2}$:

2. all its poles are simple except, possibly, at T=0
3. All its residues are in F_p

Examples:

1) α=1,1,1,1,$\ldots$ Thus Z(T)=$\frac{1}{1-T}$ and α has an abelian realization over any field.

2) α=1,2,3,4,5,$\ldots$ Thus Z(T)=$\frac{1}{(1-T)^2}$ and α has no abelian realization over any field

3) α=1,2,1,2,1,2,$\ldots$ Thus Z(T)=$\frac{1+2T}{1-T^2} = \frac{\frac{3}{2}}{1-T} + \frac{-\frac{1}{2}}{1+T}$, so α has an abelian realization over any field of characteristic not 2.

4) α=1,1,2,3,5,8,13,$\ldots$ Thus Z(T)=$\frac{1}{1-T-T^2}$

α has an abelian realization: p=0,11,19,29,31,41,

α has no abelian realization: p=2,3,5,7,13,17,23,37

Note that: α=-1,2,1,3,4,7,11,$\ldots$ has an abelian realization over any field.

- Remark: abelian realization in characteristic p>0 implies a realization with C=Vec. This is NOT the case in characteristic 0. Deligne categories (like Rep(S_t)) of super-exponential growth are needed.

Example:

1) α=1,2,1,2,1,2,… so Z(T)=$\frac{\frac{3}{2}}{1-T}+\frac{-\frac{1}{2}}{1+T}$; realization of α requires Rep(S_t) with t=3/2 and t=1/2.
2) α=3,1,3,1,3,1,… Here Z(T)=$\frac{2}{1-T}+\frac{1}{1+T}$; realization of α requires Rep(S_{-1}).

- Answer 4 (well known?) Assume p=0 and Z(T)=$\delta_0+\delta_1$T $+\sum_i \frac{\beta_i}{1-\gamma_i T}$

Sequence α admits an abelian realization of exponential growth if and only if δ_1 is an integer and $\beta_i\gamma_i$ (for all i) are integers >0.

This implies a realization with C=sVec.

Sequence α admits an realization with C=Vec if and only if δ_1 is an integer >1 and $\beta_i\gamma_i$ (for all i) are integers >0.

- Crucial tool: quotients by negligible morphisms (“gligible” quotients) aka “semisimplifications”.

  Theorem of Andre-Kahn (abstract version of Jannsen theorem) implies:

  Corollary: α admits an abelian realization if and only if the gligible quotient Cob_α of SCob_α is semisimple.

  Important computation: compute categories Cob_α.

- Results for α with abelian realization (k algebraically closed):

  1) if Z(T)=sum of partial fractions then the gligible quotient is a product of quotients for each summand.

  Thus if Z(T)=$\delta_0 + \delta_1$T+ $\sum_i \frac{\beta_i}{1-\gamma_i T}$ we need to consider only the following cases:

  2) If Z(T)=$\frac{\beta}{1-\gamma T}$ then the semisimplification of Cob_2(α) is semisimplification of Rep(S_t) with t=$\beta\gamma$ (exceptional values of t: non-negative integers)

  3) if Z(T)=$\delta_0 + \delta_1$T with $\delta_1 \neq 0$ then the semisimplification

of Cob_2(α) is semisimplification of Rep(O_t) with t=$\delta_1 - 2$ (exceptional values of t: integers)

4) if Z(T)=δ_0 then the semisimplification of Cob_2(α) is Rep(osp(1|2)) or, in characteristic p, a semisimplification

Example: Z(T)=1+2T. In characteristic $\neq 2$ we get C=Vec and A=k$[x]/x^2$. In characteristic 2 we get C=Z/2Z-graded vector spaces and A=k$[x]/(x^2-1)$.

$$\sum_{n\geq 0} a_n \frac{t^n}{n!} = \exp(2\exp(t)-2)$$

What about sequences without abelian realization, e.g. 1,2,3,4,5,…?

We can start by finding dimensions of Hom spaces in the gligible quotients Cob_α, e.g. Hom(1,$A^{\otimes n}$)=:a_n.

Conjecture: Assume Z(T)=$\frac{1}{(1-T)^2}$ or, more generally,

Theorem. Assume b_2 is nonzero. Then we have

Z(T)=$\frac{\beta}{(1-\gamma T)^2}$. Then the category SCob_α has no negligible morphisms, i.e. SCob_α =Cob_α . Thus the generating function

$$\sum_{n\geq 0} c_n \frac{t^n}{n!} = \exp(3t+t^2+t^3/6)$$

What if Z(T)=quadratic polynomial=b_0+b_1T+b_2T^2. Then the category SCob_α does have negligible morphisms, i.e. SCob_$\alpha \neq$ Cob_α .