A categorified double centraliser theorem and applications to Soergel bimodules

(joint work with Mackaay, Mazorchuk, Tubbenhauer and Zhang)

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Motivation

Why 2-representation theory?

Goals:

- Provide abstract methods to apply to examples of categorification in representation theory.
- Apply them to at least some of these examples.

Note: All categories in this talk are assumed to be locally small (or small if necessary). Further, $\Bbbk$ is an algebraically closed field.

Simple modules for an algebra?

- A: finite-dimensional, associative $\Bbbk$-algebra
- Decompose $1_A = \epsilon_1 + \cdots + \epsilon_n$ into primitive orthogonal idempotents.
- $\epsilon_j \sim \epsilon_k$ if ϵ_j, ϵ_k are in the same component for the Artin-Wedderburn decomposition $A/\mathrm{rad}(A) \cong \prod_{i=1}^r M_{n_i}(\Bbbk)$.
- Pick a set $\{e_1, \dots, e_r\}$ of representatives of $\{\epsilon_1, \dots, \epsilon_n\}/\sim$
- Set $S_i = Ae_i/\mathrm{rad}(Ae_i)$.

Then

- $\{\text{simple } A\text{-modules}\}/\cong \leftrightarrow \{S_i | i = 1, \dots, r\}$
- $\mathrm{End}_A(S_i) \cong e_iAe_i/\mathrm{rad}(e_iAe_i) \cong \Bbbk$.
- We have a double centraliser theorem:

$$\mathrm{End}_{\mathrm{End}_A(S_i)}(S_i) \cong A/\mathrm{ann}(S_i)$$

and $A/\mathrm{ann}(S_i) \cong M_{n_i}(\Bbbk)$ is Morita equivalent to $\mathrm{End}_A(S_i) \cong \Bbbk$.

Upshot: All of this categorifies in some form, but not on the the nose (no semisimplicity modulo radical, each 'matrix ring' analogue can have lots of simples, endomorphisms of simples can be nontrivial, etc).

2-categories

Definition. A 2-**category** $\mathscr{C}$ consists of

- a class (or set) $\mathscr{C}$ of objects;
- for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms from $\mathtt{i}$ to $\mathtt{j}$
  - objects in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 1-**morphisms** of $\mathscr{C}$,
  - morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 2-**morphisms** of $\mathscr{C}$;
- functorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity 1-morphisms $\mathbb{1}_{\mathtt{i}}$ for every $\mathtt{i} \in \mathscr{C}$;
- natural (strict) axioms.

Weak axioms yield a **bicategory**.

General examples of 2-categories

Examples.

- A (strict) monoidal category $\mathcal{C}$ is a bicategory (2-category) with one object, which has the objects of $\mathcal{C}$ as 1-morphisms, and the morphisms of $\mathcal{C}$ as 2-morphisms.

- the 2-category **Cat** of small categories (1-morphisms are functors and 2-morphisms are natural transformations);

- the 2-category $\mathfrak{A}^f_{\mathbb{k}}$ whose
  - objects are small idempotent complete $\mathbb{k}$-linear additive categories with finitely many indecomposable objects up to isomorphism and finite-dimensional morphism spaces
    (that is, equivalent to the category of finitely generated projective modules over a finite-dimensional $\mathbb{k}$-algebra);
  - 1-morphisms are additive $\mathbb{k}$-linear functors;
  - 2-morphisms are natural transformations.

Finitary 2-categories

Definition. A 2-category $\mathscr{C}$ is **finitary** over $\Bbbk$ if

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i},\mathtt{j})$ is in $\mathfrak{A}_{\Bbbk}^{f}$ (i.e. equivalent to A-proj for some algebra A);
- composition is biadditive and $\Bbbk$-bilinear;
- identity 1-morphisms are indecomposable.

Moral: Finitary 2-categories are 2-analogues of finite dimensional algebras.

Definition. A 2-category $\mathscr{C}$ is **fiat** (**f**initary - **i**nvolution - **a**djunction - **t**wo category) if

- it is finitary;
- there is a weak involutive equivalence $(-)^*\colon \mathscr{C} \to \mathscr{C}^{\mathrm{op,op}}$ such that there exist adjunction morphisms $F \circ F^* \to \mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{j}} \to F^* \circ F$.

Examples

Examples

- tensor categories (only weakly fiat)
- fusion categories (semi-simple tensor categories)
- projective endofunctors of A-mod (finitary for finite dimensional A, fiat if A weakly symmetric)
- finitary quotients of Kac–Moody 2-categories (aka KLR 2-categories, categorified quantum groups)
- Soergel bimodules (aka Hecke 2-categories)

Hecke algebras

(W, S) Coxeter group

$W = \langle s_i |, s_i \in S, s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$ for some $m_{ij} \in \{2, 3, \cdots, \infty\}$

$\mathrm{H}(W)$ **Hecke algebra**: quantisation of $\mathbb{Z}W$

Kazhdan–Lusztig basis $\rightsquigarrow$ cell theory (Kazhdan–Lusztig cells)

$\mathcal{J}$ **two-sided cell** or $\mathcal{H}$ intersection of a **left** with its dual **right cell**
$\rightsquigarrow$ asymptotic algebras $\mathrm{A}_{\mathcal{J}}$, $\mathrm{A}_{\mathcal{H}}$ $(q \to 0)$ [Lusztig]

$\exists$ bijection

$$\{\text{simple representations of the asymptotic algebras}\}$$
$$\updownarrow$$
$$\{\text{simple representations of the Hecke algebra}\}$$

Idea: Asymptotic algebras are easier to understand. They are essentially matrix algebras with a modified multiplication, but the classification of simple modules is not affected.

Soergel bimodules or the Hecke 2-category

(W, S, V) finite Coxeter system, V reflection representation

$R = \mathbb{C}[V]/(\mathbb{C}[V]^W)_+$ coinvariant algebra

$R_i := R \otimes_{R^{s_i}} R$ for $s_i \in S$

The 2-**category** $\mathscr{S} = \mathscr{S}_{W,S,V}$ **of Soergel bimodules** or **Hecke 2-category** has

- one object $\varnothing$ (identified with $R\text{-}\mathrm{grproj}$);
- 1-morphisms are endofunctors of $\varnothing$ isomorphic to tensoring with shifts of direct summands of direct sums of finite tensor products (over R) of the R_i;
- 2-morphisms are all natural transformations (bimodule morphisms).

Fact: Indecomposable 1-morphisms are labelled by elements in W, and $\mathscr{S}$ categorifies the Hecke algebra. In particular, indecomposable 1-morphism descend to a cellular basis (the KL-basis). [Soergel]

2-representations

From now on, $\mathscr{C}$ denotes a fiat 2-category.

Definition. A **finitary** 2-**representation M** of $\mathscr{C}$ is a (strict) 2-functor $\mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$, i.e.

- $\mathbf{M}(\mathtt{i}) \approx B_{\mathtt{i}}\text{-proj}$ for some algebra $B_{\mathtt{i}}$;
- for $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$, $\mathbf{M}(\mathrm{F})\colon \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{j})$ is an additive functor;
- for $\alpha\colon \mathrm{F} \to \mathrm{G}$, $\mathbf{M}(\alpha)\colon \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{G})$ is a natural transformation.

Example.

- For each object $\mathtt{i}$ in $\mathscr{C}$, we have the **principal** 2-representation $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$.
- Projective A-A-bimodules acting on $A\text{-proj}$.

Definition. **M** is **simple transitive** if $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}$-stable ideals.

Goal. Classify simple transitive 2-representations for interesting 2-categories.

Cell combinatorics for 2-categories

$\Sigma(\mathscr{C})$, the set of isoclasses of indecomposable 1-morphisms in $\mathscr{C}$, has several partial preorders.

left preorder: $\mathrm{F} \geq_L \mathrm{G}$ if $\exists \mathrm{H}$ such that F is a direct summand of HG
left cells: equivalence classes w.r.t. $\geq_L$

Similarly:
right preorder: $\mathrm{F} \geq_R \mathrm{G}$ if $\exists \mathrm{H}$ such that F is a direct summand of GH
right cells: equivalence classes w.r.t. $\geq_R$

two-sided preorder: $\mathrm{F} \geq_J \mathrm{G}$ if $\exists \mathrm{H}_1, \mathrm{H}_2$ such that F is a direct summand of $\mathrm{H}_1\mathrm{G}\mathrm{H}_2$

two-sided cells: equivalence classes w.r.t. $\geq_J$

Cell combinatorics for 2-categories

Example. Cells for the 2-category $\mathscr{S}$ of Soergel bimodules are Kazhdan–Lusztig cells.

E.g. let $W = \langle s, t | s^2 = 1 = t^2, stst = tsts \rangle$ of type B_2. Cells are given by

1

s, sts	st
ts	t, tst

$stst$

An $\mathcal{H}$-**cell** is the intersection of a left and a right cell.

A two-sided cell is **strongly regular**, if every $\mathcal{H}$-cell in it has precisely one element.

Simple transitive 2-representations

Lemma. [Chan–Mazorchuk] Every simple transitive 2-representation $\mathbf{M}$ has an **apex**, which is the unique maximal two-sided cell $\mathcal{J}$ such that $\mathbf{M}(\mathcal{J}) \neq 0$.

$\rightsquigarrow$ study simple transitive 2-representations apex by apex

To each left cell $\mathcal{L}$ in $\mathscr{C}$, we can associate a **cell** 2-representation $\mathbf{C}_{\mathcal{L}}$, which is simple transitive by construction.

Theorem. [Mazorchuk–M.] If the apex of a simple transitive 2-representation $\mathbf{M}$ is strongly regular, $\mathbf{M}$ is equivalent to a cell 2-representation.

Known: Simple transitive implies cell for

- appropriate quotients of Kac–Moody 2-categories [Mazorchuk–M, Macpherson];
- 2-categories of projective bimodules [Mazorchuk–M, Mazorchuk–M–Zhang];
- Soergel bimodules in type A, but **not** in other types.

$\mathcal{H}$-cell reduction

Let $\mathcal{L} \subseteq \mathcal{J}$ be a left cell in $\mathscr{C}$ and set $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^*$.

Construct $\mathscr{C}_{\mathcal{H}}$ in several steps:

- take quotients by all two-sided cells $\mathcal{J}' \not\leq \mathcal{J}$;
- inside quotient, take additive closure of $\mathbb{1}_{\mathtt{i}(\mathcal{H})}$ and 1-morphisms in $\mathcal{H}$;
- factor out the maximal ideal not containing id_{F} for $\mathrm{F} \in \mathcal{H}$.

Theorem. [Mackaay–Mazorchuk–M–Zhang] There is a bijection

$$\{\text{simple transitive 2-representations of } \mathscr{C} \text{ with apex } \mathcal{J}\}$$
$$\updownarrow$$
$$\{\text{simple transitive 2-representations of } \mathscr{C}_{\mathcal{H}} \text{ with apex } \mathcal{H}\}$$

Upshot: concentrate on $\mathscr{C}_{\mathcal{H}} \rightsquigarrow$ smaller! We call this **$\mathcal{H}$-cell reduction**.

Double Centraliser Theorem

Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_{\mathcal{H}}$.

There is a canonical 2-functor

$$\mathrm{can}\colon \mathscr{C}_{\mathcal{H}} \to \mathscr{E}nd_{\mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}).$$

Theorem. *[Double Centraliser Theorem]*
There is an equivalence of 2-semicategories

$$\mathscr{E}nd^{inj}_{\mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}) \simeq \mathrm{add}(\mathcal{H}),$$

where ^{inj} refers to restricting to injective endofunctors.

Simple transitive 2-representations for $\mathscr{S}$?

$\mathcal{H}$-cell reduction $\rightsquigarrow$ suffices to classify simple transitive 2-representations for $\mathscr{S}_{\mathcal{H}}$, where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in each two-sided cell.

To $\mathscr{S}_{\mathcal{H}}$, associate the **asymptotic bicategory** $\mathscr{A}_{\mathcal{H}}$:

- take 2-subcategory generated by bimodules with tops in degree ≥ 0;
- quotient out by ideal generated by those with tops in degree > 0.

Facts:

- categorifies $\mathrm{A}_{\mathcal{H}}$ (not graded anymore!) [Lusztig, Elias-Williamson]
- fusion bicategory
- well-understood for almost all $\mathcal{H}$, incl. classification of simple transitive 2-representations [Ostrik et al.]

Simple transitive 2-representations of $\mathscr{S}$?

Let $\mathbf{C}_{\mathcal{H}}$ be the cell 2-representation of $\mathscr{S}_{\mathcal{H}}$ associated to $\mathcal{H}$.

Double centraliser theorem $\rightsquigarrow$ equivalence of 2-semicategories

$$\mathscr{E}nd^{inj}_{\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}})}(\mathbf{C}_{\mathcal{H}}) \simeq \mathrm{add}(\mathcal{H}).$$

Theorem. There is a biequivalence

$$\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \simeq \mathscr{A}_{\mathcal{H}}$$

Caution: Ignoring gradings here for nicer statements!

Using these results, we can show:

Theorem. There is an biequivalence of 2-categories

$\{$(graded) simple transitive 2-representations of $\mathscr{A}_{\mathcal{H}}\}$

$\updownarrow$

$\{$(graded) simple transitive 2-representations of $\mathscr{S}_{\mathcal{H}}$ with apex $\mathcal{H}\}$

Thank you!

Thank you for your attention!