Jacobson-Morozov Lemma for Lie superalgebras using semisimplification

Inna Entova-Aizenbud, Vera Serganova

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Jacobson-Morozov Lemma (classical setting)

Base field: $\mathbb{C}$.
Let $\mathbb{G}_a$ be the additive affine algebraic group, and fix $\iota : \mathbb{G}_a \subset SL_2$ an embedding of $\mathbb{G}_a$ as a maximal unipotent subgroup.

Theorem (Jacobson-Morozov Lemma for algebraic groups)

Let L be a reductive algebraic group. Given an embedding of algebraic groups $i : \mathbb{G}_a \hookrightarrow L$, there exists $\bar{i} : SL_2 \to L$ such that $\bar{i} \circ \iota = i$.
The map $\bar{i}$ is unique up to conjugation by an element of L.

Passing to Lie algebras, we have:

Theorem (Jacobson-Morozov Lemma for Lie algebras)

Given a semisimple Lie algebra $\mathfrak{l}$ and a nilpotent $x \in \mathfrak{l}$, can embed x into a Lie subalgebra $\mathfrak{sl}_2 \subset \mathfrak{l}$.
This embedding is unique up to conjugation by an element of L.

Today: do the same for Lie superalgebras!

Tensor categories

A ($\mathbb{C}$-linear) tensor category:

- A ($\mathbb{C}$-linear) abelian category $\mathcal{A}$,
- A bilinear bifunctor $- \bigotimes - : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$,
- **Monoidal**: unit object $\mathbf{1}$ s.t. $\mathbf{1} \otimes A \cong A \otimes \mathbf{1} \cong A$ $(\forall A \in \mathcal{A})$,
- **Symmetric**: $\forall A_1, A_2 \in \mathcal{A}$, fix $c_{A_1,A_2} : A_1 \otimes A_2 \xrightarrow{\sim} A_2 \otimes A_1$,
- **Rigid**: any object A has a dual A^*, with

$$coev_A : \mathbf{1} \to A \otimes A^*, \;\; ev_A : A^* \otimes A \to \mathbf{1}$$

Trace of endomorphism: $\forall f \in \mathrm{End}(A)$,

$$tr(f) \in \mathrm{End}(\mathbf{1}) : \mathbf{1} \to A \otimes A^* \xrightarrow{f \otimes \mathrm{Id}} A \otimes A^* \xrightarrow{c_{A,A^*}} A^* \otimes A \to \mathbf{1}$$

Dimension of object: $\dim(A) := tr(\mathrm{Id}_A) \in \mathrm{End}(\mathbf{1})$.
In our setting, $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$.

Most basic example: $Vect_{\mathbb{C}}$

- Unit: $\mathbf{1} := \mathbb{C}$,
- Symmetry: $c_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1, \; v_1 \otimes v_2 \mapsto v_2 \otimes v_1$
- Rigidity: for any dual bases $\{v_i\}_i, \{v_i^*\}_i$ of V, V^*,

$$coev_V : \mathbb{C} \to V \otimes V^*, \; 1 \mapsto \sum_{i=1}^{\dim V} v_i \otimes v_i^*$$

$$ev_V : V^* \otimes V \to \mathbb{C}, \; f \otimes v \mapsto f(v)$$

More general example:
$Rep(G)$ (f.d. representations) for any group G.
For affine algebraic G, $Rep(G)$ is semisimple iff G is reductive.

Vector Superspaces

Definition

$SVect_{\mathbb{C}}$: category of f.d. $\mathbb{Z}_2$-graded (="super") vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$, grading-preserving linear maps.

For homogeneous $v \in V$, $p(v) := \begin{cases} 0 \text{ if } v \in V_{\bar{0}} \\ 1 \text{ if } v \in V_{\bar{1}} \end{cases}$

- Parity shift: $V \mapsto \Pi V$, with $(\Pi V)_{\bar{p}} := V_{\overline{1-p}}$, $p = 0, 1$.
- Unit: $\mathbf{1} := (\mathbb{C})_{\bar{0}}$ (purely even),
- Symmetry: $c_{V,W} : V \otimes W \to W \otimes V,\ v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v$
- Rigidity: $V^* := V^*_{\bar{0}} \oplus V^*_{\bar{1}}$, $coev_V$, ev_V as for vector spaces.
  Integer (super)dimensions:

$$sdim\, V = \dim_{Vect} V_{\bar{0}} - \dim_{Vect} V_{\bar{1}}$$

If $V_{\bar{1}} = 0$, V behaves like a usual vector space.

Example: $\mathbb{C}^{m|n} = (\mathbb{C}^m)_{\bar{0}} \oplus (\mathbb{C}^n)_{\bar{1}}$, $m, n \geq 0$. $sdim\, \mathbb{C}^{m|n} = m - n$

Algebraic supergroups and their Lie superalgebras

In *SVect*, can define (super)commutative (super)algebras, Lie (super)algebras, Hopf (super)algebras.

Definition (Algebraic supergroup)

- Affine algebraic supergroup $G \rightsquigarrow$ Hopf (super)algebra $O(G)$.
- Alternatively, a representable functor G from the category of supercommutative algebras to *Grps*.

Definition (Lie superalgebra)

$(\mathfrak{g}, [,])$ where $\mathfrak{g}$ is a vector superspace, $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ for homogeneous $a, b \in \mathfrak{g}$.
- "Super" Jacobi identity.

Remark: $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, and $ad : \mathfrak{g}_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$ a representation.
Define $Lie(G)$ as the Lie superalgebra corresponding to $O(G)$:
$Lie(G) = (I/I^2)^*$ where $I = Ker(O(G) \to \mathbb{C})$ is the augmentation ideal.

Representations of supergroups

Denote by $Rep(G)$ the category of f.d. (algebraic) representations of G in $SVect_{\mathbb{C}}$.
This is a tensor category.

Another approach to $Rep(G)$ **(Masuoka)**:
Use Harish-Chandra pairs $(\mathfrak{g}, G_0)$. Here

1. $\mathfrak{g}$ is a f.d. Lie superalgebra,
2. G_0 algebraic group, $Lie(G_0) = \mathfrak{g}_{\bar{0}}$,
3. $G_0 \curvearrowright \mathfrak{g}_{\bar{1}}$ with differential $ad : \mathfrak{g}_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$.

Any such pair $(\mathfrak{g}, G_0)$ defines a unique algebraic supergroup G!

Then $Rep(G)$ is the category of f.d. $\mathfrak{g}$-modules, integrable over G_0.

Examples of supergroups

Example (The general linear supergroup)

$GL(m|n) := \mathrm{Aut}^\bullet(\mathbb{C}^{m|n})$.

The corresponding Lie superalgebra is

$$\mathfrak{gl}(m|n) = \mathrm{Lie}\, GL(m|n) \cong \mathrm{End}^\bullet(\mathbb{C}^{m|n}), \quad [A, B] = AB - (-1)^{p(A)p(B)} BA$$

for homogeneous $A, B \in \mathrm{End}^\bullet(\mathbb{C}^{m|n})$, and $GL(m|n)_0 = GL(m) \times GL(n)$.
The category $Rep(GL(m|n))$ is not semisimple!

Example (Supergroup $OSp(m|2n)$)

Let $V = \mathbb{C}^{m|2n}$ and ω the natural non-degenerate symmetric bilinear form on V. Then $G = OSp(m|2n) \subset GL(m|2n)$ is the supergroup preserving ω.
Here $G_0 = O(m) \times Sp(2n)$.

Representations of supergroups: $OSp(1|2)$

Example (Supergroup $OSp(1|2)$)

Lie superalgebra: $\mathfrak{osp}(1|2) := Lie(OSp(1|2))$.

$$\mathfrak{osp}(1|2)_{\bar{0}} = \mathfrak{sl}_2, \ \mathfrak{osp}(1|2)_{\bar{1}} \cong \mathbb{C}^2 =: span\{x, y\}$$

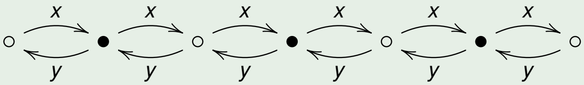
with ad given by $\mathfrak{sl}_2 \curvearrowright \mathbb{C}^2$. Harish-Chandra pair: $(\mathfrak{osp}(1|2), SL_2)$.

$Rep(OSp(1|2))$ is semisimple.

Simple modules: $\overline{M}_{2k}, \Pi\overline{M}_{2k}, k \geq 0$.

- $\dim \overline{M}_{2k} = (k+1|k)$, $sdim\, \overline{M}_{2k} = 1$.
- $\overline{M}_{2k}|_{\mathfrak{sl}_2} = V_k \oplus \Pi V_{k-1}$ ($\mathfrak{sl}_2 \curvearrowright V_k$ is irreducible, $\dim V_k = k+1$).

$\overline{M}_6$:



Example (Supergroup $\mathbb{G}_a^{(1|1)}$)

$\mathbb{G}_a^{(1|1)}$ given by the Harish-Chandra pair $(\mathfrak{g}_a^{(1|1)}, \mathbb{G}_a)$.

Lie superalgebra: $\mathfrak{g}_a^{(1|1)} := Lie(\mathbb{G}_a^{(1|1)}) = span\{[x,x], x\}$, x odd.
A $(1|1)$-dim. nilpotent Lie superalgebra,

$$\left(\mathfrak{g}_a{}^{(1|1)}\right)_{\bar{0}} = span\{[x,x]\} = \mathfrak{g}_a, \ \ [x,[x,x]] = 0$$

$\mathbb{G}_a^{(1|1)} \subset OSp(1|2)$, with $\mathfrak{g}_a^{(1|1)} \subset \mathfrak{osp}(1|2)$ a maximal nilpotent subalg.

$Rep(\mathbb{G}_a^{(1|1)})$ is not semisimple. **Indecomposables**: M_k, ΠM_k, $k \geq 0$.

- $\dim M_k = k+1$, $sdim\ M_{2k} = 1$, $sdim\ M_{2k+1} = 0$.
- $M_k = span\{v, x.v, x^2.v, \ldots, x^k.v\}$, $p(x^r.v) = r \mod 2$.

M_6 : $\quad \circ \xrightarrow{x} \bullet \xrightarrow{x} \circ \xrightarrow{x} \bullet \xrightarrow{x} \circ \xrightarrow{x} \bullet \xrightarrow{x} \circ$

Reductive and quasi-reductive supergroups

Reductive supergroups are rare:

Theorem

Let G be a connected algebraic supergroup s.t. $Rep(G)$ is semisimple. Then there exist $n_1, \ldots, n_k \in \mathbb{N}$ and a reductive algebraic group L so that

$$G \cong L \times OSp(1|2n_1) \times \cdots \times OSp(1|2n_k).$$

Instead, have useful notion:

Definition (Quasi-reductive supergroups)

A supergroup G corresponding to a Harish-Chandra pair $(\mathfrak{g}, G_0)$ is called *quasi-reductive* if G_0 is reductive.

Fact: G quasi-reductive $\Leftrightarrow$ $Rep(G)$ has enough projectives.

Example

$GL(m|n)$, $SL(m|n)$, $OSp(m|2n)$, $P(n)$, $Q(n)$ are quasi-reductive.

Overview: the new players

classical version	super version
additive group $\mathbb{G}_a$	additive supergroup $\mathbb{G}_a^{(1\|1)}$
$\mathfrak{g}_a = Lie(\mathbb{G}_a)$	$\mathfrak{g}_a^{(1\|1)} = Lie(\mathbb{G}_a^{(1\|1)})$, $\left(\mathfrak{g}_a^{(1\|1)}\right)_{\bar{0}} = \mathfrak{g}_a$
$f \in \mathfrak{g} = Lie(G)$ nilpotent $\Leftrightarrow i_f : \mathbb{G}_a \to G$	$x \in \mathfrak{g} = Lie(G)$ odd nilpotent $\Leftrightarrow i_x : \mathbb{G}_a^{(1\|1)} \to G$
algebraic group SL_2	algebraic supergroup $OSp(1\|2)$
$\mathfrak{sl}_2 = Lie(SL_2)$	$\mathfrak{osp}(1\|2) = Lie(OSp(1\|2))$, $\mathfrak{osp}(1\|2)_{\bar{0}} = \mathfrak{sl}_2$
$\mathbb{G}_a \hookrightarrow SL_2$	$\mathbb{G}_a^{(1\|1)} \hookrightarrow OSp(1\|2)$
reductive groups	quasi-reductive supergroups

Jacobson-Morozov Lemma for supergroups

Definition (Neat elements)

Nilpotent **odd** operator $x \in \mathrm{End}(V)_{\bar{1}}$ is *neat* if under the corresponding action of $\mathbb{G}_a^{(1|1)}$ on V, all the indecomposable summands of V have **non-zero** superdim.

Theorem (E.-Serganova, 2019)

Let G be a quasi-reductive supergroup, and $\mathfrak{g} = Lie(G)$. Let $x \in \mathfrak{g}, x \neq 0$ be an odd nilpotent element s.t. $x|_V$ is neat, for any $V \in Rep(G)$.
Let $i_x : \mathbb{G}_a^{(1|1)} \hookrightarrow G$ be the homomorphism corresponding to $x \in \mathfrak{g}$. Then i_x can be extended to a homomorphism $\bar{i}_x : OSp(1|2) \hookrightarrow G$, unique up to conjugation.

Remark

$x|_V$ is neat for any $V \iff x|_V$ is neat for **some** *faithful G-rep. V.*

Example

Example (Supergroup $GL(1|2)$)

$$\mathfrak{gl}(1|2) = \mathrm{End}^{\bullet}(\mathbb{C}^{1|2}) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) : \begin{smallmatrix} A \in Mat_{1\times 1}(\mathbb{C}), & B \in Mat_{1\times 2}(\mathbb{C}) \\ C \in Mat_{2\times 1}(\mathbb{C}), & D \in Mat_{2\times 2}(\mathbb{C}) \end{smallmatrix} \right\}$$

and the neat elements are:

$$\left\{ \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ d & 0 & 0 \end{pmatrix}, \begin{array}{c} ac + bd = 0, \\ (a, b) \neq (0, 0), \ (c, d) \neq (0, 0) \end{array} \right\} \cup \{0\}.$$

For $x = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right)$, the theorem gives a subalgebra $\mathfrak{osp}(1|2) \subset \mathfrak{gl}(1|2)$ generated by x and $y = \left(\begin{smallmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)$. That is,

$$\mathfrak{osp}(1|2) \cong span\{x, y, [x, y], [x, x], [y, y]\} = \left\{ \begin{pmatrix} 0 & a & b \\ -b & e & c \\ a & d & -e \end{pmatrix} \in \mathfrak{gl}(1|2) \right\}$$

Why neatness?

Recall:
- irreducible representations of $OSp(1|2)$ are $\overline{M}_{2k}$, $\Pi\overline{M}_{2k}$, $k \geq 0$.
- indecomposable representations of $\mathbb{G}_a^{(1|1)}$ are M_k, ΠM_k, $k \geq 0$.

sdim $M_k = 0$ iff k odd.

Restriction from $OSp(1|2)$ **to** $\mathbb{G}_a^{(1|1)}$: $\overline{M}_{2k} \longmapsto M_{2k}$.

Let G be a quasi-reductive supergroup, and $\mathfrak{g} = Lie(G)$.
Let $x \in \mathfrak{g}, x \neq 0$ be an odd nilpotent element and $i_x : \mathbb{G}_a^{(1|1)} \hookrightarrow G$ be the corresponding homomorphism factoring through $\mathbb{G}_a^{(1|1)} \hookrightarrow OSp(1|2)$.

Let $V \in Rep(G)$, and consider the restriction

$$V|_{OSp(1|2)} = \bigoplus_{k\geq 0} \overline{M}_{2k}^{\oplus n_k} \oplus \bigoplus_{k\geq 0} \Pi\overline{M}_{2k}^{\oplus n'_k}, \quad n_k, n'_k \in \mathbb{Z}_{\geq 0}$$

Then $V|_{\mathbb{G}_a^{(1|1)}} = \bigoplus_{k\geq 0} M_{2k}^{\oplus n_k} \oplus \bigoplus_{k\geq 0} \Pi M_{2k}^{\oplus n'_k}$, so all indecomposable summands have *sdim* $\neq 0$. **Thus $x|_V$ is necessarily neat!**

Key tool: semisimplification

Definition

The *semisimplification* of a tensor category $\mathcal{A}$ is the universal pair $(S, \overline{\mathcal{A}})$ such that $\overline{\mathcal{A}}$ is a semisimple tensor category and $S : \mathcal{A} \to \overline{\mathcal{A}}$ is a full $\otimes$-functor.

Explicit construction: (for reasonable tensor categories)
Consider the set of "negligible morphisms" in $\mathcal{A}$:

$$\mathcal{N} = \{f : A_1 \to A_2, f \in Mor(\mathcal{A}) : \quad \forall g : A_2 \to A_1,\ tr(g \circ f) = 0\}$$

It forms an ideal under composition and tensor product.
$\overline{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ is a semisimple tensor category, the quotient functor $S : \mathcal{A} \to \overline{\mathcal{A}}$ is a $\otimes$-functor, and $(S, \overline{\mathcal{A}})$ is the semisimplification of $\mathcal{A}$.

Example (André-Kahn-O'Sullivan)

Let $\mathcal{A} := Rep(\mathbb{G}_a)$. Then $\overline{\mathcal{A}} \cong Rep(SL_2)$ and the quotient functor S satisfies: $S \circ Res^{SL_2}_{\mathbb{G}_a} = \mathrm{Id}$.

Main theorem: idea of proof

Proposition (E.-Serganova, 2019)

The semisimplification of $Rep(\mathbb{G}_a^{(1|1)})$ is $(S, Rep(OSp(1|2)))$, where $S : Rep(\mathbb{G}_a^{(1|1)}) \to Rep(OSp(1|2))$ is a full $\otimes$-functor with

$$S \circ Res^{OSp(1|2)}_{\mathbb{G}_a^{(1|1)}} = \mathrm{Id}, \text{ and } S(M_k) = \begin{cases} \overline{M}_k & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}.$$

As in the Theorem, let G, $\mathfrak{g} = Lie(G)$, $x \in \mathfrak{g}_{\bar{1}} \setminus \{0\}$ be nilpotent and neat, and $i_x : \mathbb{G}_a^{(1|1)} \hookrightarrow G$. Have a restriction functor $R_x := Res^G_{\mathbb{G}_a^{(1|1)}}$:

$$\begin{array}{ccc} Rep(G) & \xrightarrow{R_x} & Rep(\mathbb{G}_a^{(1|1)}) \\ & \searrow^{S \circ R_x} & \downarrow S \\ & & Rep(OSp(1|2)) \end{array}$$

$x|_V$ is neat for all $V \in Rep(G)$ so $S \circ R_x$ is faithful $\Longrightarrow$ $S \circ R_x$ is an exact $\otimes$-functor $\Longrightarrow$ $S \circ R_x = Res^G_{OSp(1|2)}$ for some $\bar{i}_x : OSp(1|2) \hookrightarrow G$.

Further results

Let G be quasi-reductive.

Proposition (E.-Serganova)

The set of neat nilpotent elements of $\mathfrak{g}_{\bar{1}}$ has finitely many G_0-orbits.

Proposition (E.-Serganova)

If all elements in $\mathfrak{g}_{\bar{1}}$ are neat then $Rep(G)$ is semisimple.

Further results

What about non-neat odd elements $x \in \mathfrak{g}$?

For any nilpotent $x \in \mathfrak{g}_{\bar{1}}$, we obtain a similar diagram

$$\begin{array}{ccc} Rep(G) & \xrightarrow{R_x} & Rep(\mathbb{G}_a^{(1|1)}) \\ & \searrow^{S \circ R_x} & \downarrow S \\ & & Rep(OSp(1|2)) \end{array}$$

but now $S \circ R_x$ is a non-exact $\otimes$-functor.

Example: if $x^2 = 0$, we obtain the Duflo-Serganova functor DS_x, with $OSp(1|2)$ acting trivially on any $DS_x(M) = S \circ R_x(M)$.

Study $M \in Rep(G) \mapsto Supp(M) = \{x : S \circ R_x(M) \neq 0\}$**.**

Thank you!