Skein Categories

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December 3, 2020

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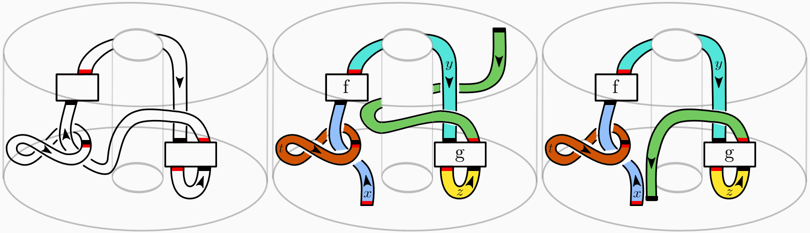


Skein algebra $\rightarrow$ Skein categories $\rightarrow$ Internal skein algebras

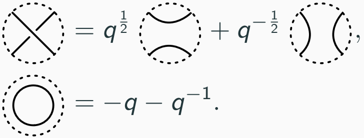
$\downarrow$

Factorisation homology

$$\begin{array}{ccc} \mathbf{Disc}_2^{or,\sqcup} & \xrightarrow{F_{\mathcal{V}}} & \mathbf{Cat}_k^{\times} \\ \big\downarrow & \nearrow_{\mathbf{Sk}_{\mathcal{V}}(\Sigma)} & \\ \mathbf{Mfld}_2^{or,\sqcup} & & \end{array}$$

Skein Algebras

The Kauffman bracket skein algebra $\mathrm{SkAlg}_q(\Sigma)$ of the oriented smooth surface Σ is the $\mathbb{Q}(q)$ module of formal linear combinations of links up to isotopy modulo the Kauffman bracket skein relations



Multiplication is given by stacking.

It is an invariant of framed links and it can be renormalised to give the Jones polynomial.

Coloured Ribbon Graphs

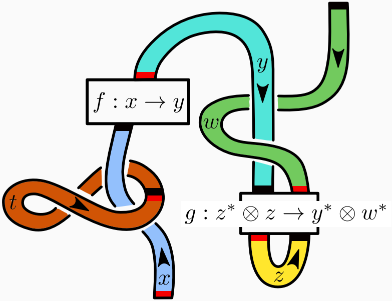
Let $\mathcal{V}$ be a (strict) ribbon category:

monoidal product $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$

braiding $\beta : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \times \mathcal{V}$

twist $\theta : \mathcal{V} \to \mathcal{V}$

duals X^* with unit $\eta : 1 \to V^* \otimes V$ and counit $\epsilon : V \otimes V^* \to 1$ maps



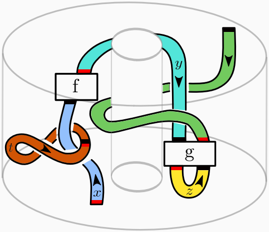
Category of Coloured Ribbons

A coloured ribbon diagram of the surface Σ is an embedding of a ribbon graph into $\Sigma \times [0,1]$ such that unattached bases are sent to $\Sigma \times \{0,1\}$.

$\mathbf{Ribbon}_{\mathcal{V}}(\Sigma)$ is the k-linear category:

Objects Finite collections of disjoint, framed, coloured, directed points in Σ

Morphisms Finite k-linear combinations of coloured ribbon diagram which are compatible which the points attached to up to isotopy which preserves ribbon graph structure.

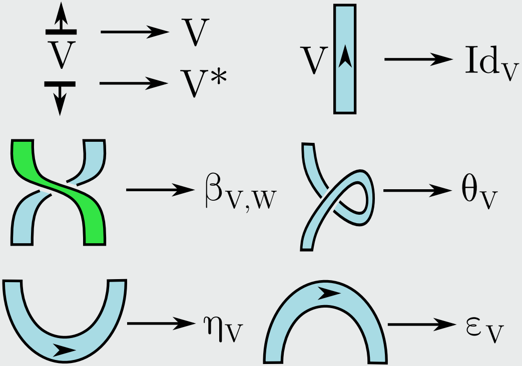


Evaluation Function

Theorem (Turaev)

There is a full surjective ribbon functor

$$\text{eval} : \mathbf{Ribbon}_{\mathcal{V}}([0,1]^3) \to \mathcal{V}$$



Skein Category (Walker, Johnson-Freyd)

The skein category $\mathbf{Sk}_{\mathcal{V}}(\Sigma)$ is the k-linear category of coloured ribbons $\mathbf{Ribbon}_{\mathcal{V}}(\Sigma)$ modulo the following relation on morphisms

$$\sum_i \lambda_i F_i \sim 0$$

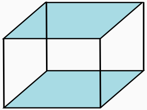
if there exists an orientation preserving embedding

$$E : [0,1]^3 \hookrightarrow \Sigma \times [0,1]$$

such that

$$\text{eval}\left(\sum_i \lambda_i F_i|_{[0,1]^3}\right) = 0,$$

the coloured ribbon diagrams F_i are identical outside the cube, and they only intersect the cube with strands transversely at the top and the bottom.



Framed E_n-Algebras

A framed E_n-algebra is symmetric monoidal functor

$$F : \mathbf{Disc}_n^{or,\sqcup} \to \mathcal{C}^{\otimes} : F(\mathbb{D}) = \mathcal{V}$$

$\mathcal{C}^{\otimes}$ is an $(\infty, 1)$ monoidal category.

$\mathbf{Mfld}_n^{or,\sqcup}$ is the symmetric monoidal $(\infty, 1)$-category:

Objects smooth oriented n-dimensional manifolds

Morphisms ∞-groupoid $\mathrm{Emb}_n^{or}(\_, \_)$

$\mathbf{Disc}_n^{or,\sqcup}$ is the full subcategory of finite disjoint unions of $\mathbb{R}$

Factorisation Homology (Lurie, Ayala, Francis, Tanaka)

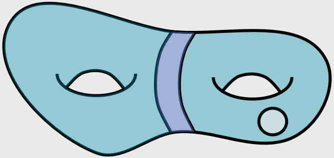
The factorisation homology $\int_{\_} \mathcal{V}$ is the Left Kan extension

$$\begin{array}{ccc} \mathbf{Disc}_n^{or,\sqcup} & \xrightarrow{F} & \mathcal{C}^{\otimes} \\ \big\downarrow & \nearrow & \\ \mathbf{Mfld}_n^{or,\sqcup} & & \end{array}$$

Theorem

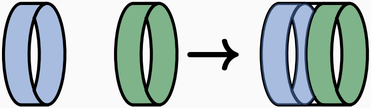
There is an equivalence of categories

$$\int M \sqcup_A N \simeq \int_M \mathcal{V} \otimes_{\int_A \mathcal{V}} \int_N \mathcal{V}$$

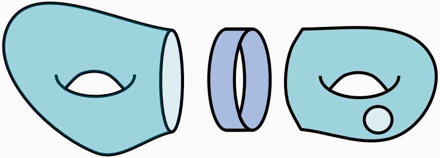


Relative Tensor Product

Let $A = \gamma \times [0,1]$. Then $\mathcal{A} := \int_A \mathcal{V}$ is a monoidal category



Assume there are actions $\int_M \mathcal{V} \curvearrowleft \mathcal{A} \curvearrowright \int_N \mathcal{V}$



The relative tensor product $\int_M \mathcal{V} \otimes_{\int_A \mathcal{V}} \int_N \mathcal{V}$ is the colimit in $\mathcal{C}^{\otimes}$ of the 2-sided bar construction:

$$\cdots \int_M \mathcal{V} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \int_N \mathcal{V} \rightrightarrows \int_M \mathcal{V} \otimes \mathcal{A} \otimes \int_N \mathcal{V} \rightrightarrows \int_M \mathcal{V} \otimes \int_N \mathcal{V}$$

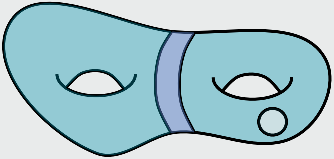
Characterisation

Theorem (Ayala, Francis, Tanaka)

The functor $\int_{-} \mathcal{V} : \mathbf{Mfld}_2^{or,\sqcup} \to \mathcal{C}^{\otimes}$ *is characterised by*

1. $\int_U \mathcal{V} \simeq \mathcal{V}$ *if* U *is contractible*
2. *if* $A \cong Y \times \mathbb{R}$ *for* 1*-manifold with corners* Y *then* $\int_A \mathcal{V}$ *has canonical monoidal structure (which does not depend on choice of homeomorphism)*
3. *Excision*

$$\int M \sqcup_A N \simeq \int_M \mathcal{V} \otimes_{\int_A \mathcal{V}} \int_N \mathcal{V}$$



Skein Categories as Factorisation Homology

$\mathrm{Sk}_{\mathcal{V}}(\Sigma)$ is a small k-linear category

An oriented embedding of surfaces $\Sigma \hookrightarrow \Pi$ induces a k-linear functor $\mathrm{Sk}_{\mathcal{V}}(\Sigma) \to \mathrm{Sk}_{\mathcal{V}}(\Pi)$,

So have a 2-functor

$$\mathrm{Sk}_{\mathcal{V}}(\_) : \mathbf{Mfld}_2^{or,\sqcup} \to \mathbf{Cat}_k^{\times}$$

$\mathbf{Cat}_k^{\times}$ is the symmetric monoidal $(2,1)$-category:

Objects small k-linear categories.

1-morphisms k-linear functors

2-morphisms natural transformations

Skein Categories as Factorisation Homology

Theorem (C.)

The skein category functor is the factorisation homology

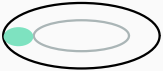
$$\begin{array}{ccc} \mathbf{Disc}_2^{or,\sqcup} & \xrightarrow{F_{\mathcal{V}}} & \mathbf{Cat}_k^{\times} \\ \big\downarrow & \nearrow_{\mathrm{Sk}_{\mathcal{V}}(\_)} & \\ \mathbf{Mfld}_2^{or,\sqcup} & & \end{array}$$

Using characterisation of factorisation homology the hard part is proving excison:

1. The relative tensor product $\mathrm{Sk}_{\mathcal{V}}(M) \times_{\mathrm{Sk}_{\mathcal{V}}(A)} \mathrm{Sk}_{\mathcal{V}}(N)$ defined as the colimit of the 2-sided bar construction is equivalent to the standard relative tensor product of k-linear categories.
2. Using this relative tensor product $\mathrm{Sk}_{\mathcal{V}}(M) \times_{\mathrm{Sk}_{\mathcal{V}}(A)} \mathrm{Sk}_{\mathcal{V}}(N) \simeq \mathrm{Sk}_{\mathcal{V}}(M \sqcup_A N)$.

Internal Skein Algebras (Gunningham, Jordan, Safronov)

The internal skein algebra of the punctured surface Σ^* is the functor



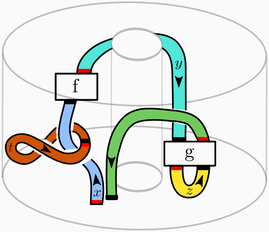
$$\mathrm{SkAlg}^{int}_{\mathcal{V}}(\Sigma^*) : \mathcal{V}^{op} \to \mathbf{Vect}$$

$$V \mapsto \mathrm{Hom}_{\mathbf{Sk}_{\mathcal{V}}(\Sigma^*)}(\mathcal{P}(V), 1)$$

Such a functor is an object in the free-cocompletion

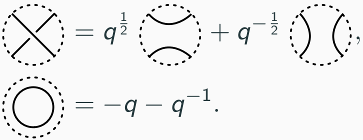
$$\overline{\mathcal{V}} := \mathbf{PSh}(\mathcal{V}^{op}, \mathbf{Vect}) \in \mathbf{LFP}_k.$$

It is an algebra internal to $\mathbf{LFP}_k$.

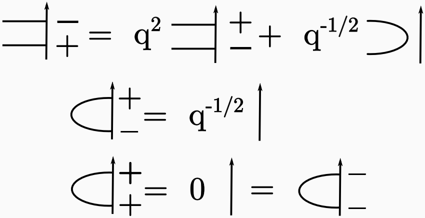


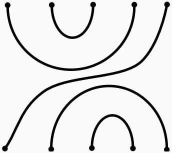
Stated Skein Algebras (Lê)

The Stated skein algebra $\mathrm{StatedSkAlg}_q(\Sigma^*)$ of the punctured oriented smooth surface Σ^* is the $\mathbb{Q}(q)$ module of formal linear combinations of stated boundary tangles up to isotopy modulo the Kauffman bracket skein relatios



and

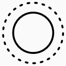




The Temperly-Lieb category **TL** is a ribbon category

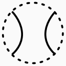
objects $[n]$ for $n \in \mathbb{Z}_{\geq 0}$

morphisms Temperley-Lieb diagrams modulo linear relation

 $= -q - q^{-1}$

composition of morphism is given by vertical stacking

monoidal product is given by horizontal stacking

braiding $\beta_{[1],[1]} :=$  $= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$ 

duality caps and cups

twist $\theta_{[1]} = -q^{-3}$

Relation to Stated Skein Algebras

From now on we shall assume q is generic.

In this case, the Cauchy completion $\mathbf{TL}$ is the category $\mathbf{Rep}_q^{fd}(\mathrm{SL})$ of finite dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$.

Theorem (Gunningham, Jordan, Safronv)

$\hat{T}(\mathrm{SkAlg}_{\mathbf{TL}}^{int}(\Sigma^*))$ *is the stated skein algebra* $\mathrm{StatedSkAlg}_q(\Sigma^*)$.

Have a functor $F : \mathbf{TL} \to \mathbf{Rep}^{fd}(\mathrm{SL}_2)$ sending [1] to standard 2-dimensional representation V. Compose with the forgetful functor to get a functor $F : \mathbf{TL} \to \mathbf{Vect}$ then $\hat{T} : \hat{\mathbf{TL}} \to \mathbf{Vect}$ is unique colimit preserving extension of this.

Reflection Equation Algebras (Majid)

Let R be the R-matrix of the quantum group $\mathcal{U}_q(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra associated to the Lie group G.

The reflection equation algebra $\mathcal{O}_q(G)$ has generators $\mathrm{u} = \{u^i_j\}$ statisfying the reflection equation

$$R_{21}\mathrm{u}_1 R\mathrm{u}_2 = \mathrm{u}_2 R_{21}\mathrm{u}_1 R$$

where $\mathrm{u}_1 := \mathrm{u} \otimes 1$ and $\mathrm{u}_2 := 1 \otimes \mathrm{u}$ and braided versions of the relations associated to G e.g. for $G = \mathrm{SL}_2$ we quotient by the braided determinant $\underline{\mathrm{det}}(\mathrm{u}) := u^1_1 u^2_2 - q^2 u^2_1 u^1_2$ is 1.

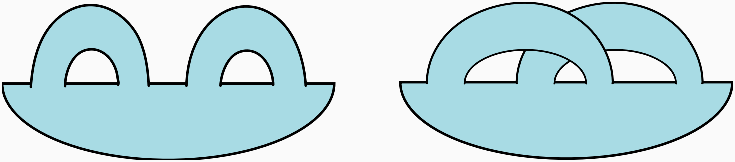
Alekseev Moduli Algebra

Theorem (Ben-Zvi, Brochier, Jordan)

The internal skein algebra $\mathrm{SkAlg}^{int}_{\mathbf{Rep}_q^{fd}(\mathrm{SL}_2)}(\Sigma^*)$ *is isomorphic to the Alekseev moduli algebra*

$$A_{\Sigma^*} = \underbrace{\mathcal{O}_q(G)\tilde{\otimes}\ldots\tilde{\otimes}\mathcal{O}_q(G)}_{\text{number of handles}}$$

where $\tilde{\otimes}$ *depends on the type of the handle.*



David Ayala, John Francis, and Hiro Lee Tanaka. "Factorization homology of stratified spaces". In: *Selecta Math. (N.S.)* 23.1 (2017), pp. 293–362.

A. Yu. Alekseev. "Integrability in the Hamiltonian Chern-Simons theory". In: *Algebra i Analiz* 6.2 (1994), pp. 53–66.

D. Ben-Zvi, A. Brochier, and D. Jordan. "Integrating quantum groups over surfaces". In: *J. Topol.* 11.4 (2018), pp. 873–916.

Francesco Costantino and Thang T. Q. Le. "Stated skein algebras of surfaces". arXiv:1912.02440. 2019.

Juliet Cooke. "Excision of Skein Categories and Factorisation Homology". In: (2019). arXiv: 1910.02630.

Selected References ii

- Sam Gunningham, David Jordan, and Pavel Safronov. "The finiteness conjecture for skein modules". arXiv:1908.05233. 2019.
- Theo Johnson-Freyd. *Heisenberg-picture quantum field theory*. 2015. eprint: `arXiv:1508.05908`.
- Shahn Majid. *Foundations of Quantum Group Theory*. Cambridge University Press, 1995.
- Peter Tingley. "A minus sign that used to annoy me but now I know why it is there (two constructions of the Jones polynomial)". In: vol. 46. Proc. Centre Math. Appl. Austral. Nat. Univ. Austral. Nat. Univ., Canberra, 2017, pp. 415–427.
- Vladimir G. Turaev. *Quantum Invariants of Knots and 3-manifolds*. de Gruyter, 1994.

Questions?