Notes for a mini-course on "Mixed Tate Motives and Fundamental Groups" given in Bonn

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September 12, 2016

Abstract

In the first part we review representation theorems for triangulated categories of Tate motives in terms of modules over Adams graded E_∞-algebras. In the second part we show how these algebras can be used to define affine derived fundamental groups for Tate motives. In the third part we apply this theory to geometric fundamental groups. Throughout we motivate our constructions with topological examples.

Contents

1 Introduction

These notes are an expanded version of a mini-course at Bonn on "Mixed Tate Motives and Fundamental Groups" given 14th-16th of September 2016.

We reviewed constructions of triangulated and abelian categories of mixed Tate motives over base schemes and applied a Tannakian formalism to those in particular cases. We put our emphasis on the passage from classical Tannakian fundamental groups to derived fundamental groups which cover more cases of categories of mixed Tate motives.

Acknowledgements: I would like to thank Peter Arndt, Hongyi Chu, Hadrian Heine, Manfred Stelzer and Sean Tilson for very helpful discussions and suggestions on the subject. I thank the HIM Bonn for providing me the opportunity to give these lectures and the organizers of the Workshop "Interactions between operads and motives" to invite me.

2 First examples for representation theorems

Tate motives on a scheme X can be thought of as motivic sheaves on X which can be constructed from the simple building blocks $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, by certain operations like taking extensions or cones, sums, etc.

In many instances such geometrically defined categories have descriptions in terms of algebraic objects.

Example 2.1. *Local systems of k-modules (k a field or ring) on a connected manifold are equivalent to representations in k-modules of the fundamental group (after picking a base point).*

We will be interested in motivic analogues of this observation.

2.1 Sheaves on a topological space

Example 2.1 is about the first homotopy group of a space, we will now look at categories for which also higher homotopy groups will play a role, but to keep things simple our local systems will become in some sense unipotent (or ind-versions thereof).

To be more specific, for a topological space X and commutative ring R we let $\mathsf{D}(X,R)$ be the derived category of sheaves of R-modules on X and $\mathsf{D}^{\mathrm{un}}(X,R)$ the full localizing subcategory of $\mathsf{D}(X,R)$ generated by the tensor unit R (note $\mathsf{D}(X,R)$ is a tensor triangulated category).

Proposition 2.2. *Let X be a manifold which is of the homotopy type of a finite CW-complex. Then there is an equivalence of tensor triangulated categories*

$$\mathsf{D}^{\mathrm{un}}(X,R) \simeq \mathsf{D}(H(X,R)),$$

where $H(X,R)$ is the function spectrum $(HR)^{\Sigma^\infty_+ X}$ viewed as an E_∞-ring spectrum.

Sketch of proof. For a topological space X let $\mathsf{D}^{\mathrm{lc}}(X,R)$ be the full triangulated subcategory of $\mathsf{D}(X,R)$ of locally constant objects, i.e. of such $F \in \mathsf{D}(X,R)$ such that there exists an open cover $X = \bigcup_i U_i$ such that for every i the restriction $F|_{U_i}$ lies in the image of the canonical functor $\mathsf{D}(R) \to \mathsf{D}(U_i,R)$. If we assume that every open of X is paracompact and that X is Hausdorff and locally contractible then one can derive an equivalence

$$\mathsf{D}^{\mathrm{lc}}(X,R) \simeq \mathrm{Fun}(X,\mathsf{D}(R))$$

of symmetric monoidal ∞-categories (here we view X as an ∞-groupoid and $\mathsf{D}(\cdots)$ also denotes the ∞-categorical versions of the derived categories). (Choose a contractible hypercover (the geometric realization of its connected components will be equivalent to X), use descent for the sheaf categories and the analogous statement for contractible opens $U \subset X$.)

Now for X as in the Proposition the functor $X \to *$ induces a symmetric monoidal functor

$$\mathsf{D}(R) \to \mathrm{Fun}(X,\mathsf{D}(R)).$$

Since $H(X,R)$ is the image of the tensor unit under the right adjoint of this functor we get an induced left adjoint

$$\mathsf{D}(H(X,R)) \to \mathrm{Fun}(X,\mathsf{D}(R)).$$

This functor is a full embedding, since it induces equivalences on mapping spaces between shifts of the tensor units, $\mathsf{D}(H(X,R))$ is compactly generated by the tensor unit and the right adjoint of this functor preserves filtered colimits, since X is finite. The essential image can then be identified with $\mathsf{D}^{\mathrm{un}}(X,R)$. □

Remark 2.3. *Note that in general $\mathsf{D}(X,R)$ is far from being compactly generated (if X is a non-compact connected manifold of dimension ≥ 1 then the only compact objects in $\mathsf{D}(X,\mathbb{Z})$ are the zero objects, see [30]).*

Remark 2.4. *In general a local system on a space X with values in an ∞-category C is a functor $X \to C$. If X is path-connected and $x \in X$ is a base point, then $X \simeq B\Omega X$, so that a functor $B\Omega X \simeq X \to C$ is a representation of the group-like A_∞-space ΩX in C, which generalizes Example 2.1. If $C = \mathsf{Spc}$ is the ∞-category of spaces and $\omega_x\colon \mathrm{Fun}(X,\mathsf{Spc}) \to \mathsf{Spc}$ denotes the fiber functor at x, it is shown in [40] that ΩX can be recovered as the endomorphism space of ω_x.*

2.2 Representations of a prounipotent group scheme

Let $G = \mathrm{Spec}(A)$ be an affine group scheme over a field k and

$$\mathrm{Rep}(G) \simeq \mathrm{Comod}^{\mathrm{fd}}(A)$$

its category of finite dimensional representations (i.e. finite dimensional A-comodules). Then $\mathrm{Ind}(\mathrm{Rep}(G))$ is the category of all A-comodules, and there is a finitely generated symmetric monoidal model structure on $\mathrm{Cpx}(\mathrm{Ind}(\mathrm{Rep}(G)))$ (the category of complexes of A-comodules), where the weak equivalences are the homotopy isomorphisms, see [17, Theorem 2.5.17], [17, Proposition 4.2.14].

We let $\mathrm{Rep}_\infty(G)$ be the associated symmetric monoidal ∞-category, which is compactly generated presentable and stable. We have a canonical symmetric monoidal left adoint

$$\mathsf{D}(k) \to \mathrm{Rep}_\infty(G).$$

We denote by $H(G)$ the image of the tensor unit under the right adjoint of this functor.

Proposition 2.5. *Let the notation be as above. Then the induced symmetric monoidal left adoint*

$$\mathrm{Mod}(H(G)) \to \mathrm{Rep}_\infty(G)$$

is a full embedding. If G is prounipotent it is an equivalence. (Here we call G prounipotent if the only positive-dimensional irreducible representation of G is the trivial representation on the one-dimensional vector space k.)

Sketch of proof. The first assertion follows from the fact that this functor induces equivalences between mapping spaces between shifts of the tensor units, that the source category is compactly generated by the tensor unit and that the tensor unit in the target category is as well compact. The second assertion follows from the fact that then the tensor unit also generates the target category. □

Remark 2.6. *If G is prounipotent we will see in Remark 7.9 that G can be reconstructed from the E_∞-algebra $H(G)$ in $\mathsf{D}(k)$.*

3 Motivic categories

A base scheme will always mean a separated Noetherian scheme of finite Krull dimension.

The unstable motivic category $\mathcal{H}(S)$ of a base scheme S is defined to be the $\mathbb{A}^1$-localization of $\mathrm{Sh}(\mathrm{Sm}_{S,Nis}, \mathsf{Spc})$, where $\mathrm{Sm}_{S,Nis}$ is the site of all smooth separated S-schemes of finite type with the Nisnevich topology. The $(\mathbb{P}^1)$-stable motivic category of S, $\mathsf{SH}(S)$, is the $\mathbb{P}^1$-stabilzation of $\mathcal{H}_\bullet(S)$.

$\mathsf{SH}(S)$ is a compactly generated presentable symmetric monoidal stable ∞-category.

For E a commutative monoid object in $\mathsf{SH}(\mathrm{Spec}(\mathbb{Z}))$ we denote by $\mathsf{SH}_E(S)$ the category $\mathrm{Mod}(f^*E)$, where $f\colon S \to \mathrm{Spec}(\mathbb{Z})$ is the structure morphism.

The assignment $S \mapsto \mathsf{SH}_E(S)$ satisfies the 6-functor formalism ([2]) except possibly absolute purity (here we denote by $\mathsf{SH}_E(S)$ also the homotopy category of the ∞-category defined above).

In particular for $E = \mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ the motivic Eilenberg-MacLane spectrum over $\mathrm{Spec}(\mathbb{Z})$ as constructed in [35] and a base scheme X we let

$$\mathrm{DM}(X) := \mathsf{SH}_{\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}}(X) = \mathrm{Mod}(\mathsf{M}\mathbb{Z}_X).$$

We denote by $\mathbb{Z}(1)$ the object $\mathsf{M}\mathbb{Z}_X \wedge \Sigma^\infty(\mathbb{P}^1, \{\infty\})[-2] \in \mathrm{DM}(X)$ and let $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \in \mathbb{Z}$ (note $\mathbb{Z}(1)$ is tensor-invertible in $\mathrm{DM}(X)$).

There is a canonical functor $D(\mathbb{Z}) \to \mathrm{DM}(X)$. In particular for any abelian group we have the Tate objects $A(n) := \mathbb{Z}(n) \otimes A$, $n \in \mathbb{Z}$.

Definition 3.1. *The category of (triangulated) Tate motives (or Tate sheaves)* $\mathrm{DMT}(X)$ *over* X *is defined to be the full localizing triangulated subcategory of* $\mathrm{DM}(X)$ *generated by the* $\mathbb{Z}(n)$, $n \in \mathbb{Z}$.

We denote by $\mathrm{DM}_{\mathrm{gm}}(X)$ the full triangulated subcategory of $\mathrm{DM}(X)$ of compact objects, and similarly for $\mathrm{DMT}_{\mathrm{gm}}(X)$.

We define motivic cohomology of X with coefficients in an abelian group A as

$$H^p(X, A(q)) \coloneqq \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}, A(q)[p]),$$

$p, q \in \mathbb{Z}$.

We have the following properties of motivic cohomology:

Theorem 3.2. *If* X *is ind-smooth over a Dedekind domain (here a field also counts as a Dedekind domain) then*

1. $H^p(X, A(q)) = 0$ *for* $q < 0$, $p \in \mathbb{Z}$,
2. $H^0(X, A(0)) = A^{\pi_0(X)}$,
3. $H^p(X, A(0)) = 0$ *for* $p \neq 0$,
4. $H^1(X, \mathbb{Z}(1)) = \mathcal{O}_X^*(X)$,
5. $H^2(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$,
6. $H^p(X, \mathbb{Z}(1)) = 0$ *for* $p \neq 1, 2$,
7. $H^p(X, A(q)) = 0$ *for* $p > q$ *and* X *the spectrum of a local ring,*
8. $H^p(X, A(q)) = 0$ *for* $p > q + \dim(X)$.

For any base scheme X we can extract from $\mathsf{M}\mathbb{Z}_X$ motivic complexes

$$\mathcal{M}(n)^X \in \mathsf{D}(\mathrm{Sh}(X_{Zar}, \mathbb{Z})),$$

$n \in \mathbb{Z}$, such that for every $U \in X_{Zar}$ we have $H^p(U, \mathbb{Z}(q)) \cong H_{Zar}(U, \mathcal{M}(q)^X)$ (here the objects of X_{Zar} are étale schemes over X). Item (7) in Theorem 3.2 says that for X ind-smooth over a Dedekind domain we have $\mathcal{H}^p(\mathcal{M}(q)^X) = 0$ for $p > q$.

We denote by ϵ the map of sites $X_{\acute{e}t} \to X_{Zar}$.

Theorem 3.3. *If* X *is ind-smooth over a Dedekind domain and* $m \in \mathbb{N}_{>0}$ *invertible on* X *then there is a canonical isomorphism*

$$\mathcal{M}(n)^X/m \cong \tau_{\leq n}(\mathbb{R}\epsilon_* \mu_m^{\otimes n})$$

in $\mathsf{D}(\mathrm{Sh}(X_{Zar}, \mathbb{Z}/m))$ *for any* $n \in \mathbb{N}$.

Proof. This follows from [13] since the Bloch-Kato conjecture is proved ([41]). □

We say that a general base scheme X *satisfies the Beilinson-Soulé vanishing conjecture* if items (1)-(3) for $A = \mathbb{Z}$ in Theorem 3.2 hold and if for every $q > 0$ we have $H^p(X, \mathbb{Z}(q)) = 0$ for $p \leq 0$.

Lemma 3.4. *If X is ind-smooth over a Dedekind domain then X satisfies the BS-conjecture if and only if for every $q > 1$ we have $H^p(X, \mathbb{Q}(q)) = 0$ for $p \le 0$.*

Proof. This follows from the Beilinson-Lichtenbaum conjecture (Theorem 3.3), see also [23, Lemma 24], and a description of mod-p motivic cohomology for smooth schemes over fields of characteristic p (see [14]). □

Example 3.5. *Any localization of a number ring satisfies the BS-conjecture, a product of schemes of the following types over a localization of a number ring satisfies the BS-conjecture: $\mathbb{A}^1$, $\mathbb{G}_m$, $\mathcal{M}_{0,n}$ ($n \ge 3$) (open moduli space of genus 0 curves with n marked points), $\overline{\mathcal{M}}_{0,n}$ ($n \ge 3$) (Deligne-Mumford compactifications of the latter moduli spaces), Grassmannians, etc. Smooth curves over finite fields satisfy the BS-conjecture.*

Example: Motivic cohomology $H^p(\mathrm{Spec}(\mathbb{Z}), \mathbb{Z}(q))$:

q (weight) \ p (degree)	0	1	2	3	4	5
16	0	$\mathbb{Z}/16320$	$\mathbb{Z}/7234$	0	$\mathbb{Z}/2$	0
15	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	(0?)	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
14	0	$\mathbb{Z}/24$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
13	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	(0?)	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
12	0	$\mathbb{Z}/65520$	$\mathbb{Z}/1382$	0	$\mathbb{Z}/2$	0
11	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	(0?)	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
10	0	$\mathbb{Z}/264$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
9	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	(0?)	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
8	0	$\mathbb{Z}/480$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
7	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	(0?)	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
6	0	$\mathbb{Z}/504$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
5	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	(0?)	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
4	0	$\mathbb{Z}/240$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
3	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	0
2	0	$\mathbb{Z}/24$	$\mathbb{Z}/2$	0	0	0
1	0	$\mathbb{Z}/2$	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0

4 The first representation theorem for Tate motives

For a base scheme X the category $\mathrm{DMT}(X)$ is not generated by the tensor unit (in an appropriate sense) (unless X is empty), so we cannot hope to model $\mathrm{DMT}(X)$ in $\mathsf{D}(\mathbb{Z})$ as we did it in similar situations in Propositions 2.2 and 2.5.

But the category of additionally graded (one also says Adams graded) complexes of abelian groups $\mathrm{Cpx}(\mathbb{Z})^{\mathbb{Z}}$ looks like a promising candidate to be an environment for models of $\mathrm{DMT}(X)$, since it is generated (as a cocomplete stable ∞-category) by the objects $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, where this time $\mathbb{Z}(n)$ is the complex $\mathbb{Z}$ sitting in Adams degree n.

Notation 4.1. *For an ∞-category C and an Adams graded object $M \in C^{\mathbb{Z}}$ we denote by $M\{n\} \in C$ the object sitting in Adams degree $n \in \mathbb{Z}$.*

In fact we will see that there exists a commutative algebra object A in $\mathsf{D}(\mathbb{Z})^{\mathbb{Z}}$ (we write again $\mathsf{D}(\cdots)$ instead of $\mathrm{Cpx}(\cdots)$), where $\mathsf{D}(\mathbb{Z})^{\mathbb{Z}}$ is equipped with the convolution symmetric monoidal structure (the symmetric monoidal structure on the disrete category $\mathbb{Z}$ is the addition), such that

$$\mathrm{DMT}(X) \simeq \mathrm{Mod}(A)$$

as symmetric monoidal ∞-categories.

To achieve this, we have the

Proposition 4.2. *The commutative algebra $\mathsf{M}\mathbb{Z}_X \in \mathsf{SH}(X)$ is periodizable in the sense that there exists a commutative algebra $\mathsf{PM}\mathbb{Z}_X \in \mathsf{SH}(X)^{\mathbb{Z}}$ together with a canonical equivalence $\mathsf{M}\mathbb{Z}_X \simeq \mathsf{PM}\mathbb{Z}_X\{0\}$ as commutative algebras in $\mathsf{SH}(X)$ and a canonical equivalence $\mathbb{Z}(1) \simeq \mathsf{PM}\mathbb{Z}_X\{1\}$ in $\mathrm{DM}(X)$ such that the multiplication maps $\mathsf{PM}\mathbb{Z}_X\{k\} \wedge_{\mathsf{M}\mathbb{Z}_X} \mathsf{PM}\mathbb{Z}_X\{l\} \to \mathsf{PM}\mathbb{Z}_X\{k+l\}$ are equivalences for all $k, l \in \mathbb{Z}$.*

Remark 4.3. *So $\mathsf{PM}\mathbb{Z}_X \simeq \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}(n)$, and the right hand side has a graded commutative algebra structure.*

Sketch of proof of Proposition 4.2. There are at least two ways to construct a periodization: Either one modifies the construction of $\mathsf{M}\mathbb{Z}$ to directly get $\mathsf{PM}\mathbb{Z}$, see [35, Section 8], or one uses the fact that MGL has a periodization PMGL with respect to the $(2,1)$-shift ([37]) and the pushout of the canonical E_∞-orientation $\mathsf{MGL} \to \mathsf{M}\mathbb{Z}$ along $\mathsf{MGL} \to \mathsf{PMGL}$. (The last two constructions give periodizations of the form $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}(n)[2n]$, but such a periodization can be transformed into one of the required type.) □

Remark 4.4. *It is known that the sphere spectrum in SH is not periodizable with respect to the 2-shift, i.e. $\bigvee_{n\in\mathbb{Z}} S^{2n}$ cannot be given a natural E_∞-structure, even if it is a commutative monoid in the homotopy category.*

We let $c\colon \mathsf{D}(\mathbb{Z})^{\mathbb{Z}} \to \mathrm{DM}(X)^{\mathbb{Z}}$ be the prolonguation to Adams graded objects of the canonical functor $\mathsf{D}(\mathbb{Z}) \to \mathrm{DM}(X)$ and $\mathcal{A}_X \in \mathrm{Comm}(\mathsf{D}(\mathbb{Z})^{\mathbb{Z}})$ the image of $\mathsf{PM}\mathbb{Z}_X$ under the right adjoint of c.

Theorem 4.5. *For any base scheme X there is a canonical symmetric monoidal equivalence*

$$\mathrm{DMT}(X) \simeq \mathrm{Mod}(\mathcal{A}_X).$$

A similar statement holds with R-coefficients for any commutative ring R ($\mathcal{A}_X$ has then to be replaced with $\mathcal{A}_X \otimes R$).

Sketch of proof. There is a naturally induced symmetric monoidal left adjoint

$$\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(\mathsf{PM}\mathbb{Z}_X).$$

Evaluating at Adams degree 0 induces a lax symmetric monoidal functor

$$\mathrm{Mod}(\mathsf{PM}\mathbb{Z}_X) \to \mathrm{DM}(X).$$

Then one shows that composing the two functors induces a symmetric monoidal full embedding

$$\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{DM}(X)$$

whose essential image is $\mathrm{DMT}(X)$ (use that this functor induces equivalences between mapping spaces between the generating objects $\mathbb{Z}(n)[i]$, $n, i \in \mathbb{Z}$). □

Remark 4.6. *There is also a representation theorem for cellular motivic spectra: The category $\mathsf{SH}_{\mathcal{T}}(X)$ of cellular motivic spectra is the full localizing stable subcategory of $\mathsf{SH}(X)$ generated by the spheres $S^{0,n}$, $n \in \mathbb{Z}$. Then one can show that there exists a commutative algebra $\mathcal{B}_X \in \mathrm{Sp}^{QS^0}$ (where QS^0 denotes the underlying group-like E_∞-space of the sphere spectrum) together with a symmetric monoidal equivalence*

$$\mathrm{Mod}(\mathcal{B}_X) \simeq \mathsf{SH}_{\mathcal{T}}(X)$$

(this result is due to Hadrian Heine). A similar statement holds for the module category over any E_∞-ring spectrum in $\mathsf{SH}(X)$. In Theorem 4.5 one can replace QS^0 by $\pi_0(QS^0) \simeq \mathbb{Z}$ due to the good behavior of the Σ_n-action on $\mathbb{Z}(1)^{\otimes n}$, $n \in \mathbb{N}$.

Remark 4.7. *By construction the cohomology of $\mathcal{A}_X$ is the motivic cohomology of X (similarly the homotopy groups of $\mathcal{B}_X$ are the bigraded homotopy groups of the motivic sphere spectrum over X).*

So if X is ind-smooth over a Dedekind domain and m invertible on X, the structure of $\mathcal{A}_X/m$ is closely related to étale cohomology (Theorem 3.3). For example $H^0(\mathcal{A}_X/p)$ (p a prime invertible on X) is a graded polynomial algebra over $\mathbb{F}_p$ with a generator in positive Adams degree.

Remark 4.8. *There are other constructions of motivic dga's, e.g. due to Joshua. With rational coefficients one can use cycle cdga's for certain base schemes (see [26]).*

Definition 4.9. *Let R be a commutative ring and $A \in \mathrm{Comm}(\mathsf{D}(R)^{\mathbb{Z}})$. We say A is of* Tate-type *if the unit $R \to A\{0\}$ is an equivalence and if $A\{n\} \simeq 0$ for $n > 0$ (in order that our $\mathcal{A}_X$ from above is an example in favourable cases we have to flip the grading, which we will do from now on).*

Lemma 4.10. *An algebra A of Tate-type has a canonical augmentation $A \to R$ in $\mathrm{Comm}(\mathsf{D}(R)^{\mathbb{Z}})$.*

5 Tannakian categories of mixed Tate motives

According to the motivic philosophy for a field k the category $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ should possess the so-called motivic t-structure, whose heart would then be the abelian category of geometric mixed motives (with rational ceofficients) over k. This abelian category should be a Tannakian category over $\mathbb{Q}$ (possibly non-neutral). So far this t-structure is out of reach (implying in particular the BS-conjecture for smooth schemes over k). Therefore we are interested in certain subcategories for which we might be able to construct interesting t-structures.

In this direction we have the following (see [25])

Theorem 5.1 (Levine)**.** *Suppose X is a base scheme satisfying the rational BS-conjecture. Then $\mathrm{DMT}_{\mathrm{gm}}(X)_{\mathbb{Q}}$ has a natural t-structure such that the $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, all lie in the heart $\mathrm{MT}_{\mathrm{gm}}(X)_{\mathbb{Q}}$. If X is additionally connected this heart is a neutral Tannakian category over $\mathbb{Q}$ with a canonical fiber functor (taking weight graded pieces).*

Remark 5.2. *The corresponding Tannakian fundamental group G_X is an extension of $\mathbb{G}_{m,\mathbb{Q}}$ by a prounipotent group scheme U_X over $\mathbb{Q}$.*

Example 5.3. $\mathrm{Lie}(U_{\mathrm{Spec}(\mathbb{Z})})$ *is a free graded pro-Lie algebra on generators in degrees* $3, 5, 7, \ldots$, *see [15].*

By now there are many constructions of so-called geometric motivic fundamental groups (e.g. [9], [11], [10], [27]), in realizations and Tate motives over the base.

They arise via the following picture:

Given a diagram of base schemes

$$\begin{array}{c} X \\ p \downarrow \; \uparrow s \\ S \end{array}$$

with $p \circ s = \mathrm{id}$ such that both X and S are connected and satisfy the BS-conjecture we get a split exact sequence of group schemes

$$1 \longrightarrow K \longrightarrow G_X \underset{s_*}{\overset{p_*}{\rightleftarrows}} G_S \longrightarrow 1$$

promoting the kernel K to an affine group scheme object in $\mathrm{Ind}(\mathrm{MT}_{\mathrm{gm}}(S)_{\mathbb{Q}}) \simeq \mathrm{MT}(S)_{\mathbb{Q}}$, which is then defined to be the geometric fundamental group of X/S at s. By general Tannakian theory we have that

$$\mathrm{Rep}_{\mathrm{MT}_{\mathrm{gm}}(S)_{\mathbb{Q}}}(K) \simeq \mathrm{MT}_{\mathrm{gm}}(X)_{\mathbb{Q}},$$

where the left category is understood to be the category of representations of K inside $\mathrm{MT}_{\mathrm{gm}}(S)_{\mathbb{Q}}$.

Remark 5.4. *One version of the construction of geometric fundamental groups involves the cosimplicial path scheme of* X *over* S *(see e.g. [43]), and it was first seen using these methods that e.g. the Malčev completion of fundamental groups of certain* $\mathbb{C}$*-varieties carry mixed Hodge structures.*

Our goal is to use the motivic algebra $\mathcal{A}_X$ to define (derived) fundamental groups for Tate motives. The idea to use cycle dga's to define motivic fundamental groups and abelian categories of mixed Tate motives as representations of such is already used in [6] and [24].

6 Affine derived group schemes

Our group objects will live in a category of affine derived schemes.

We fix a symmetic monoidal ∞-category $\mathcal{C}$ and set $\mathrm{Aff}_{\mathcal{C}} := \mathrm{Comm}(\mathcal{C})^{\mathrm{op}}$. Note $\mathrm{Aff}_{\mathcal{C}}$ has finite products.

Definition 6.1. *The category of affine group schemes in* $\mathcal{C}$ *is defined to be* $\mathrm{Gp}(\mathrm{Aff}_{\mathcal{C}})$, *i.e. the full subcategory of the functor category* $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Aff}_{\mathcal{C}}) \simeq \mathrm{Fun}(\Delta, \mathrm{Comm}(\mathcal{C}))^{\mathrm{op}}$ *on functors* F *which satisfy the duals of the Segal conditions, such that* $F([0])$ *is an initial algebra and such that the resulting monoid object is group-like.*

Remark 6.2. *So such a group scheme is encoded by a cosimplicial algebra $A^\bullet$ in $\mathcal{C}$ such that $\mathbf{1} \simeq A^0$, the dual Segal maps*

$$(A^1)^{\otimes n} \to A^n$$

are equivalences, and such that a group-like condition holds (one possible formulation is that for every $B \in \mathrm{Comm}(\mathcal{C})$ the simplicial space $\mathrm{map}(A^\bullet, B)$ encodes a group-like A_∞-space, but there is also a diagrammatic formulation). We refer to [29, Def. 7.2.2.1], [29, Def. 6.1.2.7] and [29, Proposition 6.1.2.6] for details.

Lemma 6.3. *Suppose $\mathrm{Aff}_{\mathcal{C}}$ has finite limits (e.g. if $\mathcal{C}$ is presentably symmetric monoidal). For $X = \mathrm{Spec}(A \to \mathbf{1}) \in \mathrm{Aff}_{\mathcal{C},\bullet}$ let $C(X)_\bullet$ be the Čech nerve of the map $* \to \mathrm{Spec}(A)$, i.e.*

$$C(X)_n \simeq * \times_X * \times_X \cdots \times_X * \simeq \mathrm{Spec}(\mathbf{1}^{\otimes_A [n]})$$

for any $n \geq 0$. Then $C(X)_\bullet$ is an affine group scheme in $\mathcal{C}$.

Remark 6.4. *In general the Čech nerve $C(U/X)_\bullet = C(f)_\bullet$ of a map $f: U \to X$ in $\mathrm{Aff}_{\mathcal{C}}$ is a* groupoid object *in $\mathrm{Aff}_{\mathcal{C}}$ (the condition that $C(f)_0$ being the terminal object is no longer sastisfied).*

Definition 6.5. *Suppose $\mathcal{C}$ is presentably symmetric monoidal. Let I be an ∞-category and $X: I^{\mathrm{op}} \to \mathrm{Aff}_{\mathcal{C}}$, $i \mapsto X_i = \mathrm{Spec}(A_i)$, a functor. We set $\mathrm{QCoh}(X) := \lim_{i \in I} \mathrm{Mod}_{\mathcal{C}}(A_i)$.*

Note $\mathrm{QCoh}(X)$ only depends on $\mathrm{colim} X$, where the colimit is taken in $\mathrm{PSh}(\mathrm{Aff}_{\mathcal{C}}, \mathsf{Spc})$.

In particular if $X_\bullet$ is a group or groupoid object in $\mathrm{Aff}_{\mathcal{C}}$ then we think of $\mathrm{QCoh}(X_\bullet)$ as a category of representations of $X_\bullet$.

For $X \in \mathrm{Aff}_{\mathcal{C},\bullet}$ we will be particularly interested in determining conditions when the natural map $\mathrm{QCoh}(X) \to \mathrm{QCoh}(C(X)_\bullet)$ (assuming its existence, e.g. when $\mathcal{C}$ is presentably symmetric monoidal) is an equivalence, at least for certain subcategories. This applies in particular to an algebra of Tate-type and its canonical augmentation.

We have the following

Proposition 6.6. *Let $\mathcal{C}$ be presentably symmetric monoidal and $f: Y \to X$ be a map in $\mathrm{Aff}_{\mathcal{C}}$ which has a section. Then the natural map*

$$\mathrm{QCoh}(X) \to \mathrm{QCoh}(C(f)_\bullet)$$

is an equivalence.

Proof. Because f has a section the augmented simplicial object $C(f)_\bullet \to X$ can be promoted to a split augmented simplicial object. Taking module categories yields a split coaugmented cosimplicial object in $\widehat{\mathsf{Cat}}_\infty$ (the ∞-category of not necessarily small ∞-categories), and it follows that the limit of the underlying cosimplicial object is equivalent to the value at the coaugmentation. □

Given a map between two augmented simplicial objects

$$\begin{array}{ccc} V_\bullet & \xrightarrow{\psi} & U_\bullet \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\varphi} & X \end{array} \tag{1}$$

in $\mathrm{Aff}_{\mathcal{C}}$ it induces (assuming $\mathcal{C}$ is presentably symmetric monoidal) a commutative square of left adjoints

$$\begin{array}{ccc}
\mathrm{QCoh}(V_\bullet) & \xleftarrow{\psi^*} & \mathrm{QCoh}(U_\bullet) \\
\uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle f^*} \\
\mathrm{QCoh}(Y) & \xleftarrow{\varphi^*} & \mathrm{QCoh}(X).
\end{array}$$

This square induces a base change natural transformation

$$\varphi^* \circ f_* \to g_* \circ \psi^*$$

of functors from $\mathrm{QCoh}(U_\bullet)$ to $\mathrm{QCoh}(Y)$.

In case the diagram (1) is cartesian we denote this natural transformation by $B(U_\bullet/X, \varphi)$.

Given a map $Y \to X$ in $\mathrm{Aff}_{\mathcal{C}}$ we can base change it along itself to the map $\mathrm{pr}_2 : Y \times_X Y \to Y$ which has the diagonal as a section, so pr_2 satisfies descent. Under certain conditions this helps to derive descent for the original map:

Proposition 6.7. *Let $\mathcal{C}$ be presentably symmetric monoidal and $f : Y \to X$ a map in $\mathrm{Aff}_{\mathcal{C}}$. Let $\mathcal{M} \subset \mathrm{QCoh}(X)$ be a full subcategory such that*

i) the endofunctor $\tilde{f}_* \circ \tilde{f}^*$ of $\mathrm{QCoh}(X)$ (where $\tilde{f} : C(f)_\bullet \to X$ is the natural map) restricts to an endofunctor α of $\mathcal{M}$,

ii) the composition $\mathcal{M} \subset \mathrm{QCoh}(X) \xrightarrow{f^*} \mathrm{QCoh}(Y)$ is conservative (i.e. detects equivalences) and

iii) for any $M \in \mathrm{QCoh}(C(f)_\bullet)$ which lies in the image of the composition $\mathcal{M} \subset \mathrm{QCoh}(X) \to \mathrm{QCoh}(C(f)_\bullet)$ the base change map $B(C(f)_\bullet/X, f)(M)$ is an equivalence.

Then the restricted unit natural transformation $\mathrm{id}_{\mathcal{M}} \to \alpha$ is an equivalence, in particular the composition $\mathcal{M} \subset \mathrm{QCoh}(X) \to \mathrm{QCoh}(C(f)_\bullet)$ is a full embedding. Moreover an object $M \in \mathrm{QCoh}(C(f)_\bullet)$ belongs to the essential image of this composition if and only if $B(C(f)_\bullet/X, f)(M)$ is an equivalence and $\tilde{f}_*(M)$ belongs to the essential image of the full inclusion $\mathcal{M} \subset \mathrm{QCoh}(X)$.

Proof. Let $M \in \mathcal{M}$. Then because of condition iii) the map

$$f^*(M \to \alpha(M))$$

identifies with the map $f^*(M) \to (g_* \circ g^*)(f^*(M))$, where g is the base change of $\tilde{f}$ to Y. But this latter map is an equivalence by Proposition 6.6. Now ii) implies that already $M \to \alpha(M)$ is an equivalence.

For the second assertion let $M \in \mathrm{QCoh}(C(f)_\bullet)$ such that $B(C(f)_\bullet/X, f)(M)$ is an equivalence and $\tilde{f}_*(M)$ belongs to the essential image of the full inclusion $\mathcal{M} \subset \mathrm{QCoh}(X)$. Then the natural map

$$(\tilde{f}^* \circ \tilde{f}_*)(M) \to M$$

is an equivalence since its pullback $N \to M'$ to $\mathrm{QCoh}(C(f')_\bullet)$ (f' the pullback of f along f) identifies with $(g^* \circ g_*)(M') \to M'$ by iii) which is an equivalence by Proposition 6.6, and this pullback is conservative. $\square$

7 Coconnective E_∞-algebras over a field

Definition 7.1. *Let k be a field. An associative algebra $A \in \mathrm{Alg}(\mathsf{D}(k))$ is called* coconnective *if $k \to H_0(A)$ is an isomorphism and if $H_i(A) = 0$ for $i > 0$.*

Lemma 7.2. *Let $A \in \mathsf{D}(k)$ be a coconnective algebra and M a left and N a right A-module. Let $m, n \in \mathbb{Z}$ and assume $H_i(M) = 0$ for $i > m$ and $H_i(N) = 0$ for $i > n$. Then $H_i(N \otimes_A M) = 0$ for $i > m+n$ and the map $H_n(N) \otimes_k H_m(M) \to H_{m+n}(N \otimes_A M)$ is injective.*

Proof. This is [28, Corollary 4.1.11]. □

Lemma 7.3. *Let $M^\bullet$ be a cosimplicial object in $\mathsf{D}(\mathbb{Z})$ such that $H_i(M^n) = 0$ for every $i > 0$ and $n \geq 0$. Then for every $n \geq 0$ the fiber F of the natural map*

$$\mathrm{Tot}(M^\bullet) \to \mathrm{Tot}(M^{\leq n})$$

(where $M^{\leq n}$ denotes the restriction of $M^\bullet$ to $\Delta_{\leq n}$) satisfies $H^i(F) = 0$ for $i \leq n$, in particular this map induces an isomorphism in i-th cohomology for every $i < n$.

Proof. This follows from the fact that for every $n > 0$ the fiber of the map

$$\mathrm{Tot}(M^{\leq n}) \to \mathrm{Tot}(M^{\leq (n-1)})$$

is a retract of $M^n[-n]$ and $\mathrm{Tot}(M^\bullet) \simeq \lim_n \mathrm{Tot}(M^{\leq n})$. □

Lemma 7.4. *Let $A \in \mathsf{D}(k)$ be a coconnective algebra and $M^\bullet$ a cosimplicial left A-module such that there is a $K \in \mathbb{Z}$ such that $H_i(M^n) = 0$ for $i > K$ and all $n \geq 0$. Let N be a right A-module such that $H_i(N) = 0$ for $i > L \in \mathbb{Z}$. Then the natural map*

$$N \otimes_A \mathrm{Tot}(M^\bullet) \to \mathrm{Tot}(N \otimes_A M^\bullet)$$

is an equivalence.

Proof. Without loss of generality we can assume that $K = L = 0$. We show that the map in question induces an isomorphism in homology. Let $n \geq 0$, then for any $i < n$ we have a commutative square

$$\begin{array}{ccc} H^i(N \otimes_A \mathrm{Tot}(M^\bullet)) & \longrightarrow & H^i(\mathrm{Tot}(N \otimes_A M^\bullet)) \\ \downarrow & & \downarrow \\ H^i(N \otimes_A \mathrm{Tot}(M^{\leq n})) & \longrightarrow & H^i(\mathrm{Tot}(N \otimes_A M^{\leq n})). \end{array}$$

The vertical maps are isomorphisms by Lemmas 7.2 and 7.3, whereas the lower horizontal map is an isomorphism since the tensor products involved commute with finite limits. □

Theorem 7.5. *Let k be a field and $X \in \mathrm{Aff}_{\mathsf{D}(k),\bullet}$ be the spectrum of an augmented coconnective commutative algebra A in $\mathsf{D}(k)$. Then there is a natural symmetric monoidal equivalence*

$$\mathrm{QCoh}^+(X) \simeq \mathrm{QCoh}^+(C(X)_\bullet)$$

between bounded above A-modules and bounded above representations of the affine group scheme $C(X)_\bullet$.

Proof. We apply Proposition 6.7 with $f = X$ and $\mathcal{M} = \mathrm{QCoh}^+(X)$. The needed truncation properties and property ii) are implied by Lemma 7.2. Property iii) is implied by Lemma 7.4. So we get a full embedding from the left category to the right. This functor is essentially surjective by the second part of Proposition 6.7. □

Remark 7.6. *Under the equivalence in Theorem 7.5 the perfect A-modules map via an equivalence to the full subcategory of* $\mathrm{QCoh}(C(X)_\bullet)$ *consisting of those cosimplicial modules* $M^\bullet$ *such that* $M^0 \in \mathsf{D}(k)$ *is perfect.*

Proof. We have to see that the induced functor between these two categories is essentially surjective. So let $M^\bullet \in \mathrm{QCoh}(C(X)_\bullet)$ be such that $M^0 \in \mathsf{D}(k)$ is perfect. We prove the statement by induction on $\sum_{i\in\mathbb{Z}} \dim_k(H_i(M^0))$. Suppose this sum is positive. Let $i \in \mathbb{Z}$ be maximal such that $H_i(M^0)$ is non-trivial. Then $H_j(\lim M^\bullet) = 0$ for $j > i$ and $H_i(\lim M^\bullet)$ is non-trivial (using Lemma 7.2). Picking a non-trivial (in homotopy) map $A[i] \to \lim M^\bullet$ and sending this map to a map f in $\mathrm{QCoh}(C(X)_\bullet)$ produces a non-trivial map $k[i] \to M^0$ in level zero. Applying the induction hypothesis to the cofiber of f shows the claim. □

Remark 7.7. *If* k *has characteristic* 0 *then the category* $\mathrm{QCoh}^+(X)$ *in Theorem 7.5 can also be identified with the bounded above quasi-coherent sheaves on the associated coaffine stack, see [28, Proposition 4.5.2. (7)].*

Proposition 7.8. *For any* E_∞*-ring spectrum* R *the stable* ∞*-category* $\mathrm{Mod}(R) \simeq \mathsf{D}(R)$ *carries a t-structure such that* $\mathsf{D}(R)_{\le 0}$ *consists of those R-modules M such that* $\pi_i M = 0$ *for* $i > 0$*, and such that* $\mathsf{D}(R)_{\ge 0}$ *is generated inside* $\mathsf{D}(R)$ *by colimits and extensions by R.*

We call the t-structure given in Proposition 7.8 the canonical t-structure.

Remark 7.9. *Let* k *be a field and* $A \in \mathrm{Comm}(\mathsf{D}(k))_{/k}$ *be an augmented coconnective commutative algebra. Let* $X = \mathrm{Spec}(A)$*. The restriction of the canonical t-structure on* $\mathrm{Mod}(A)$ *to* $\mathrm{QCoh}^+(X)$ *yields a t-structure on* $\mathrm{QCoh}^+(C(X)_\bullet)$ *(see Theorem 7.5). Its heart consists of those cosimplicial modules* $M^\bullet$ *such that the homology of* $M^0 \in \mathsf{D}(k)$ *is concentrated in degree* 0*. Such objects are in correspondence with representations of the affine k-group scheme* $U_A := \mathrm{Spec}(H^0(\Gamma(C(X)_1)))$*, where* $\Gamma(C(X)_1) \simeq k \otimes_A k$*. We see that there are tensor equivalences*

$$\mathrm{Mod}(A)^\heartsuit \simeq \mathrm{QCoh}^+(C(X)_\bullet)^\heartsuit \simeq \mathrm{Rep}(U_A).$$

From the consruction in the proof of Remark 7.6 it follows that U_A *is prounipotent. By Tannaka duality if we choose* $A := H(U)$ *(see section 2.2) for a prounipotent affine group scheme* U *over* k *we see that* H^0 *of the affine derived group scheme associated to* A *gives back* U*.*

Definition 7.10. *Let* k *be a field and* G *a discete group. We define the k-Malčev completion* $M_k(G)$ *of* G *to be the Tannaka dual of the full Tannakian subcategory of* $\mathrm{Rep}_k^{\mathrm{fd}}(G)$ *generated by the tensor unit by extensions.*

Remark 7.11. *If* G *is finitely generated then*

$$M_k(G) = \mathrm{Spec}(\mathrm{colim}_n(k[G]/I^n)^\vee),$$

where $I \subset k[G]$ *is the augmentation ideal.*

Remark 7.12. *For a pointed path-connected CW-complex (Y, y) the cohomology algebra $H(Y, k)$ (k a field) yields an object $X \in \mathrm{Aff}_{\mathsf{D}(k),\bullet}$ such that H^0 of $C(X)_\bullet$ satisfies $M_k(\pi_1(Y, y)) \cong U_{H(Y,k)}$.*

Remark 7.13. *Let R be an E_∞-ring spectrum and Y a pointed simply-connected CW-complex. Let $X := \mathrm{Spec}(H(Y, R)) \in \mathrm{Aff}_{\mathsf{D}(R),\bullet}$. Assume that the canonical map*

$$R \otimes_{H(Y,R)} R \to H(\Omega Y, R)$$

is an equivalence and that the canonical maps

$$H(\Omega Y, R)^{\otimes_R n} \to H((\Omega Y)^n, R)$$

are equvialences for all $n \in \mathbb{N}$. Then the functor

$$\mathrm{Mod}(H(Y, R))^c \to \mathrm{QCoh}(C(X)_\bullet)$$

is a full embedding. The conditions are fullfilled e.g. if k is a field and Y of finite k-type (Eilenberg-Moore spectral sequence), so we recover for this case part of Theorem 7.5. If D is a localization of the integers and Y is of finite D-type then by base changing to the field case we see that also in this case the conditions are fulfilled. Further base change to a commutative D-algebra R shows that for this R the conditions are also fulfilled. This result is joint with Sean Tilson.

Proof. We have to see that

$$\lim R^{\otimes_{H(Y,R)}[\bullet]}$$

is via the natural map equivalent to $H(R, Y)$. Using our assumptions we can rewrite this limit as

$$\lim H(*^{\times_Y [\bullet]}, R)$$

(note $*^{\times_Y n} \simeq (\Omega Y)^{n-1}$ for $n > 0$). By adjunction this is

$$\underline{\mathrm{Hom}}(\mathrm{colim}((*^{\times_Y [\bullet]}) \wedge R), R) \simeq \underline{\mathrm{Hom}}(Y \wedge R, R) \simeq H(Y, R).$$

□

8 The second representation theorem for Tate motives

Definition 8.1. *Let R be a commutative ring. An algebra $A \in \mathrm{Comm}(\mathsf{D}(R)^{\mathbb{Z}})$ of Tate-type is called* bounded *if for every $k < 0$ the complex $A\{k\}$ is cohomologically bounded from below.*

Our main result in this section is

Theorem 8.2. *Let R be a commutative ring of finite homological dimension. Let $A \in \mathrm{Comm}(\mathsf{D}(R)^{\mathbb{Z}})$ be an algebra of bounded Tate-type. Let $R \to S$ be a map in $\mathrm{Comm}(\mathsf{D}(R)^{\mathbb{Z}})$ and set $X := \mathrm{Spec}(A \otimes_R S \to S) \in \mathrm{Aff}_{\mathsf{D}(S),\bullet}$ (where we use the canonical augmentation given by Lemma 4.10). Then the adjunction*

$$\mathrm{QCoh}(X) \rightleftharpoons \mathrm{QCoh}(C(X)_\bullet$$

restricts to an equivalence

$$\mathrm{QCoh}(X)^{+}_{\mathrm{Aba}} \simeq \mathrm{QCoh}(C(X)_\bullet)^{+}_{\mathrm{Aba}}$$

(here Aba *stands for Adams bounded from above) which inherits tensor structures if S has finite homological dimension, where these subcategories consist of those modules M (resp. cosimplicial modules $M^\bullet$), such that $M\{k\} \simeq 0$ for k large enough (resp. $M^0\{k\} \simeq 0$ for k large enough) and each complex $M\{k\}$ (resp. $M^0\{k\}$) ($k \in \mathbb{Z}$) is cohomologically bounded from below.*

Idea of proof. In principle one has to apply Proposition 6.7, but it involves work to check that the assumptions there are satisfied. For details we refer to [36]. □

Definition 8.3. *For a commutative ring R and a flat affine group scheme G over R we denote by $\widetilde{G}_\bullet \in \mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(R)})$ the corresponding affine derived group scheme over R.*

Proposition 8.4. *For a commutative ring R we have*

$$\mathrm{QCoh}(\widetilde{\mathbb{G}}_{m,R,\bullet}) \simeq \mathsf{D}(R)^{\mathbb{Z}}.$$

Corollary 8.5. *i) To give an affine derived group scheme $H_\bullet \in \mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(R)^{\mathbb{Z}}})$ in $\mathsf{D}(R)^{\mathbb{Z}}$ (R a commutative ring) is the same as to give a group scheme $H'_\bullet \in \mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(R)})$ in $\mathsf{D}(R)$ together with a $\widetilde{\mathbb{G}}_{m,R,\bullet}$-action on $H'_\bullet$.*

ii) Let $H_\bullet$ and $H'_\bullet$ be as in i), then for the semi-direct product $H'_\bullet \rtimes \widetilde{\mathbb{G}}_{m,R,\bullet}$ (which is defined as the diagonal of a certain bisimplicial object in $\mathrm{Aff}_{\mathsf{D}(R)}$) we have

$$\mathrm{QCoh}(H'_\bullet \rtimes \widetilde{\mathbb{G}}_{m,R,\bullet}) \simeq \mathrm{QCoh}(H_\bullet).$$

Definition 8.6. *For a base scheme X and commutative coefficient ring R we write $\mathfrak{Gal}_{\mathrm{MT},R}(X)$ for the affine derived group scheme*

$$C(\mathrm{Spec}(\mathcal{A}_X \otimes R \to R))'_\bullet \rtimes \widetilde{\mathbb{G}}_{m,R,\bullet}$$

in the case $\mathcal{A}_X$ is of Tate-type.

Remark 8.7. *For a map $R \to S$ of commutative algebras we have*

$$\mathfrak{Gal}_{\mathrm{MT},S}(X) \simeq \mathfrak{Gal}_{\mathrm{MT},R}(X) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S).$$

Definition 8.8. *Let $G_\bullet \in \mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(R)})$ (R a commutative ring or ring spectrum), then we write $\mathrm{Rep}(G_\bullet)$ for the full subcategory of $\mathrm{QCoh}(G_\bullet)$ consisting of those cosimplicial modules $M^\bullet$ such that M^0 is a perfect R-module.*

Theorem 8.9. *Let X be connected ind-smooth over a Dedekind domain and assume X satisfies for each $q > 1$*

$$H^p(X, \mathbb{Q}(q)) = 0 \text{ for } p \ll 0$$

(a weak version of the BS-conjecture). Then we have for any commutative ring R a symmetric monoidal equivalence

$$\mathrm{DMT}_{\mathrm{gm}}(X)_R \simeq \mathrm{Rep}(\mathfrak{Gal}_{\mathrm{MT},R}(X)).$$

Remark 8.10. *This applies in particular if X satisfies the BS-conjecture, see the examples in Example 3.5.*

With finite coefficients we can do better:

Theorem 8.11. *Let X be connected ind-smooth over a Dedekind domain. Then for any $m > 0$ we have a symmetric monoidal equivalence*

$$\mathrm{DMT}_{\mathrm{gm}}(X)_{\mathbb{Z}/m} \simeq \mathrm{Rep}(\mathfrak{Gal}_{\mathrm{MT},\mathbb{Z}/m}(X)).$$

(An analogous statement holds for any finite commutative coefficient ring.)

Proof. If $m = p^r$ for a prime p we can use that with $R := \mathbb{Z}_p$ the algebra $\lim_n A_X/p^n$ is of bounded Tate type, hence we may apply Theorem 8.2. □

Remark 8.12. *Suppose X is connected and satisfies the BS-conjecture. Then the affine group scheme $Spec(H^0(\Gamma(\mathfrak{Gal}_{\mathrm{MT},\mathbb{Q}}(X))))$ over $\mathbb{Q}$ is canonically isomorphic to G_X as defined in Remark 5.2. There is a natural map*

$$\mathfrak{Gal}_{\mathrm{MT},\mathbb{Q}}(X) \to G_X.$$

This map being an equivalence exhibits a certain $K(\pi,1)$-property of X, which is fulfilled e.g. if X is a product of $\mathbb{G}_m$'s and $\mathcal{M}_{0,n}$'s ($n \geq 3$) over a localization of a number ring. This map induces a map

$$\mathsf{D}^b(\mathrm{MT}_{\mathrm{gm}}(X)_{\mathbb{Q}}) \to \mathrm{DMT}_{\mathrm{gm}}(X)_{\mathbb{Q}}.$$

Example 8.13. *Let $p \neq l$ be two prime numbers and $q = p^n$ for some $n > 0$. Let $X := \mathrm{Spec}(\mathbb{F}_q)$. Since the non-trivial motivic cohomology of X is concentrated in cohomological degree 1 (namely $H^1(X, \mathbb{Z}(m)) \cong \mathbb{Z}/(q^m - 1)$ for $m > 0$) it follows that $\Gamma(\mathfrak{Gal}_{\mathrm{MT},\mathbb{Z}}(X))$ and $\Gamma(\mathfrak{Gal}_{\mathrm{MT},\mathbb{F}_l}(X))$ are connective, so that the spectra G and G' of their H_0 are group schemes (in the first case maybe non-flat over $\mathbb{Z}$). In particular we exhibit a map*

$$G' \to \mathfrak{Gal}_{\mathrm{MT},\mathbb{F}_l}(X).$$

9 Geometric fundamental groups

We return to the situation

$$\begin{array}{c} X \\ p \big\downarrow \big\uparrow s \\ S \end{array}$$

with $p \circ s = \mathrm{id}$ from section 5 and assume that both $\mathcal{A}_S$ and $\mathcal{A}_X$ are algebras of Tate-type (in $\mathsf{D}(\mathbb{Z})^{\mathbb{Z}}$). The map p induces a map $p^* : \mathcal{A}_S \to \mathcal{A}_X$ in $\mathrm{Comm}(\mathsf{D}(\mathbb{Z})^{\mathbb{Z}})$ and the map s an augmentation in $\mathcal{A}_S$-algebras of $\mathcal{A}_X$, thus we arrive at an object

$$\mathcal{S}(X/S, s) := \mathrm{Spec}(\mathcal{A}_X \to \mathcal{A}_S) \in \mathrm{Aff}_{\mathsf{D}(\mathcal{A}_S),\bullet} \simeq \mathrm{Aff}_{\mathrm{DMT}(S),\bullet}.$$

Definition 9.1. *We set*

$$\pi_1^{\mathrm{mot}}(X/S, s) := C(\mathcal{S}(X/S, s))_\bullet \in \mathrm{Gp}(\mathrm{Aff}_{\mathrm{DMT}(S),\bullet})$$

and use also the same notation for the underlying object in $\mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(\mathbb{Z})^{\mathbb{Z}}})$ or in $\mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(\mathbb{Z})})$. We denote the base change of $\pi_1^{\mathrm{mot}}(X/S, s)$ to a commutative ring R by $\pi_1^{\mathrm{mot}}(X/S, s)_R$.

By construction we have

Proposition 9.2. *Let the situation be as above and R be a commutative ring.*

i) There is a split (by s_*) short exact sequence

$$1 \to \pi_1^{\mathrm{mot}}(X/S, s)_R \to \mathfrak{Gal}_{\mathrm{MT},R}(X) \to \mathfrak{Gal}_{\mathrm{MT},R}(S) \to 1$$

in $\mathrm{Gp}(\mathrm{Aff}_{\mathrm{D}(R)})$.

ii) Suppose S and X satisfy the assumptions of Theorem 8.9. Then we have a symmetric monoidal equivalence

$$\mathrm{DMT}_{\mathrm{gm}}(X)_R \simeq \mathrm{Rep}_{\mathrm{DMT}_{\mathrm{gm}}(S)_R}(\pi_1^{\mathrm{mot}}(X/S, s)_R).$$

Remark 9.3. *We can apply the functor*

$$p_*: \mathrm{DM}(X) \to \mathrm{DM}(S)$$

to the tensor unit arriving at an algebra $p_*(\mathbb{Z}_X) \in \mathrm{Comm}(\mathrm{DM}(S))$. Then if we view $\mathcal{A}_X$ as object of $\mathrm{DMT}(S)$ we can identify $\mathcal{A}_X$ with the image of $p_*\mathbb{Z}_X$ under the right adjoint (the Tate projection) of the full inclusion $\mathrm{DMT}(S) \hookrightarrow \mathrm{DM}(S)$. In particular, if $p_*\mathbb{Z}_X$ is already in $\mathrm{DMT}(S)$ (i.e. is a (big triangulated) mixed Tate motive) then we have an equivalence $\mathcal{A}_X \simeq p_*\mathbb{Z}_X$ in $\mathrm{DMT}(S)$ (i.e. $\mathcal{A}_X$ is the cohomological motive of X relative to S).

In the example $X = \mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0, 1, \infty\}$ and $S = \mathrm{Spec}(\mathbb{Z})$ the structure morphism $X \to S$ has no section, but we can use tangential base points instead:

Lemma 9.4. *Let X and S be as above. Every tangential base point*

$$t \in \{\overrightarrow{01}, \overrightarrow{10}, \overrightarrow{0\infty}, \overrightarrow{\infty 0}, \overrightarrow{1\infty}, \overrightarrow{\infty 1}\}$$

induces an augmentation of the $\mathcal{A}_S$-algebra $\mathcal{A}_X$. The same holds true more generally for tangential base points of scheme morphisms.

Thus we can define the geometric fundamental group

$$\pi_1^{\mathrm{mot}}((\mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0, 1, \infty\})/\mathrm{Spec}(\mathbb{Z}), \overrightarrow{01}) \in \mathrm{Gp}(\mathrm{Aff}_{\mathrm{DMT}(\mathrm{Spec}(\mathbb{Z}))})$$

and similarly other geometric fundamental groups with tangential base points as base points.

Definition 9.5. *For any base scheme X and commutative ring R we call the t-structure on $\mathrm{DMT}(X)_R$ generated by the Tate objects $R(n)$, $n \in \mathbb{Z}$, the* canonical *t-structure. We denote its heart by $\mathrm{MT}(X)_R$. This is a finitely presentable abelian closed tensor category. We denote the full subcategory of $\mathrm{MT}(X)_R$ of finitely presentable objects by $\mathrm{MT}_{\mathrm{gm}}(X)_R$ which is a closed tensor category (but we do not know in general if it is abelian).*

Remark 9.6. *If X satisfies the BS-conjecture then the Tate objects $R(n)$ actually belong to $\mathrm{MT}_{\mathrm{gm}}(X)_R$.*

Theorem 9.7. *Let X be a product of $\mathbb{G}_m$'s and $\mathcal{M}_{0,n}$'s ($n \geq 3$) over the spectrum S of a localization of a number ring and t a (tangential) base point for X/S.*

i) The group scheme $\pi_1^{\mathrm{mot}}(X/S,t) \in \mathrm{Gp}(\mathrm{Aff}_{\mathrm{DMT}(S)})$ *actually is underived, i.e. lies in* $\mathrm{Gp}(\mathrm{Aff}_{\mathrm{MT}(S)})$.

ii) Let D *be a Dedekind ring of mixed characteristic and* R *a commutative* D*-algebra which is of finite type as module over* D*. Then we have*

$$\mathrm{MT}_{\mathrm{gm}}(X)_R \simeq \mathrm{Rep}_{\mathrm{MT}_{\mathrm{gm}}(S)_R}(\pi_1^{\mathrm{mot}}(X/S,t)_R)$$

as abelian closed tensor categories.

iii) The Betti realization of $\pi_1^{\mathrm{mot}}(X/S,t)$ *is*

$$\mathrm{Spec}(\mathrm{colim}_n(\mathbb{Z}[\pi_1(X(\mathbb{C}),t)]/I^n)^\vee),$$

which is a flat group scheme over $\mathrm{Spec}(\mathbb{Z})$*, whose function algebra can be identified as those functions on* $M_{\mathbb{Q}}(\pi_1(X(\mathbb{C}),t))$ *which take integral values on integral points.*

Remark 9.8. *Realizing the fundamental group*

$$\pi_1^{\mathrm{mot}}((\mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0,1,\infty\})/\mathrm{Spec}(\mathbb{Z}), \overrightarrow{01}) \in \mathrm{Gp}(\mathrm{Aff}_{\mathrm{MT}(\mathrm{Spec}(\mathbb{Z}))})$$

gives rise to the group scheme in realizations defined in [9] (though the notion of integral structure is somewhat different).

Example 9.9. *The function Hopf algebra* $A \in \mathrm{MT}(\mathrm{Spec}(\mathbb{Z}))$ *of the geometric fundamental group* $\pi_1^{\mathrm{mot}}(\mathbb{G}_m/\mathrm{Spec}(\mathbb{Z}),1)$ *has a filtration by* A_n*'s with*

$$A_n/A_{n-1} \cong \mathbb{Z}(-n),$$

$n \geq 0$*, which does not split (it splits rationally).*

Concerning t-structures on compact objects we have the

Theorem 9.10. *Let* D *be a Dedekind domain of mixed characteristic and* X *a base scheme which satisfies the BS-conjecture. Then the canonical* t*-structure on* $\mathrm{DMT}(X)_D$ *restricts to a* t*-structure on* $\mathrm{DMT}_{\mathrm{gm}}(X)_D$*, in particular* $\mathrm{MT}_{\mathrm{gm}}(X)_D$ *is abelian.*

10 The Deligne torsors

Let $X := \mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0,1,\infty\}$, $G := \pi_1^{\mathrm{mot}}(X,\overrightarrow{01})$, $K := \mathrm{Ker}(G \xrightarrow{p} \pi_1^{\mathrm{mot}}(\mathbb{G}_m,1))$,
$U_1 := K^{\mathrm{ab}} = K/K', U := G/K'$, thus there is an exact sequence

$$1 \to U_1 \to U \to \pi_1^{\mathrm{geom}}(\mathbb{G}_m,1) \to 1.$$

Let $T := T(X,\overrightarrow{01},\overrightarrow{10})$ be the G-torsor of motivic paths from $\overrightarrow{01}$ to $\overrightarrow{10}$,

then p_*T is canonically trivialized which defines a K-torsor whose push forward along $K \to K^{\mathrm{ab}} = U_1$ we denote by T_1.

Let $A \in \mathrm{MT}(\mathrm{Spec}(\mathbb{Z}))$ be the function algebra on $\pi_1^{\mathrm{mot}}(\mathbb{G}_m,1)$,

U_1 "looks like" $A^\vee(1)$ (the latter we consider as a pro-object), so the U_1-torsor T_1 defines an extension

$$0 \to A^\vee(1) \to E \to \mathbb{Z}(0) \to 0.$$

Push forward along certain maps $A^\vee(1) \to \mathbb{Z}(n)$ defines Deligne's $\mathbb{Z}(n)$-torsors, $n \geq 1$, corresponding to extensions

$$0 \to \mathbb{Z}(n) \to E_n \to \mathbb{Z}(0) \to 0.$$

Theorem 10.1. *i) For even n the class of E_n in $H^1(\mathrm{Spec}(\mathbb{Z}), \mathbb{Z}(n))$ is a generator.*

ii) The image of the class of E_n in

$$H^1(\mathrm{Spec}(\mathbb{Z}), \mathbb{Z}(n))/\mathrm{tors} \cong \mathbb{Z}$$

is non-zero for all odd n, $n \geq 3$, and is a generator for all such n if and only if Vandiver's conjecture holds (for all primes).

Proof. Item i) follows from Deligne's result ([9]) on the order of the class of E_n in realizations (which is the denominator of $\frac{1}{2}\zeta(1-n) = -\frac{B_n}{2n}$), ii) follows from results by Ichimura-Sakaguchi on the Soulé elements ([18]). □

Remark 10.2. *By a result of Brown ([7]) geometric parts of the function algebra of $T(X, \overrightarrow{01}, \overrightarrow{10})_{\mathbb{Q}} \in \mathrm{Aff}_{\mathrm{MT}(\mathrm{Spec}(\mathbb{Z}))_{\mathbb{Q}}}$ generate $\mathrm{MT}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{Z}))_{\mathbb{Q}}$ as Tannakian category, one could ask if geometric parts of the function algebra of $T(X, \overrightarrow{01}, \overrightarrow{10}) \in \mathrm{Aff}_{\mathrm{MT}(\mathrm{Spec}(\mathbb{Z}))}$ and duals generate $\mathrm{MT}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{Z}))$ as a tensor abelian category.*

11 Outlook

Here we mainly explain connections to other works.

11.1 Tannakization (Iwanari's work)

We start with the following result ([22, Theorem 4.15]):

Theorem 11.1. *Let be given a small symmetric monoidal ∞-category C, a commutative ring spectrum R and a symmetric monoidal functor $\varphi: C \to \mathsf{D}(R)^c$. For any commutative R-algebra A let φ_A be the symmetric monoidal composition*

$$C \to \mathsf{D}(R)^c \to \mathsf{D}(A)^c.$$

Then the functor $\mathrm{Comm}_R \to \mathrm{Gp}(\mathsf{Spc})$, $A \mapsto \mathrm{Aut}(\varphi_A)$, is representable by an affine derived group scheme $\underline{\mathrm{Aut}}(\varphi)_\bullet \in \mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(R)})$, and the functor φ factors as

$$C \to \mathrm{Rep}(\underline{\mathrm{Aut}}(\varphi)_\bullet) \to \mathsf{D}(R)^c.$$

This factorization is universal for factorizations via representations of affine derived group schemes in $\mathsf{D}(R)$.

The group scheme $\underline{\mathrm{Aut}}(\varphi)_\bullet$ is called the tannakization of φ.

Corollary 11.2. *Let X satisfy the conditions of Theorem 8.9. Then for any commutative coefficient ring R the group scheme $\mathfrak{Gal}_{\mathrm{MT},R}(X)$ in $\mathsf{D}(R)$ is the tannakization of $\mathrm{DMT}_{\mathrm{gm}}(X)_R \to \mathsf{D}(R)^c$ (the latter functor is induced by the natural augmentation of $\mathcal{A}_X$).*

Proof. This follows from Theorems 8.9 and 11.1. □

Remark 11.3. *Assume X is connected and ind-smooth over a Dedekind domain. With rational coefficients it follows from [21] that the tannakization of* $\mathrm{DMT}_{\mathrm{gm}}(X)_{\mathbb{Q}} \to \mathsf{D}(\mathbb{Q})^c$ *is given by* $\mathfrak{Gal}_{\mathrm{MT},\mathbb{Q}}(X)$ *(without assuming any version of the BS-conjecture). So even if we do not know that* $\mathrm{DMT}_{\mathrm{gm}}(X)_{\mathbb{Q}}$ *is modelled as representations of* $\mathfrak{Gal}_{\mathrm{MT},\mathbb{Q}}(X)$ *(but compare with [36, Theorem 2.1]), the group scheme* $\mathfrak{Gal}_{\mathrm{MT},\mathbb{Q}}(X)$ *satisfies a universality property.*

Remark 11.4. *Let* $\mathcal{A} \in \mathrm{Comm}(\mathsf{D}(R)^{\mathbb{Z}})$*. Then we can view* $\mathcal{A}$ *also as object of* $\mathrm{Aff}_{\mathsf{D}(R)}$ *equipped with a* $\widetilde{\mathbb{G}}_{m,R,\bullet}$*-action. In Iwanari's work the quotien stack* $[\mathrm{Spec}(\mathcal{A})/\widetilde{\mathbb{G}}_{m,R,\bullet}] \in \mathrm{PSh}(\mathrm{Aff}_{\mathsf{D}(R)}, \mathsf{Spc})$ *is used, and it holds*

$$\mathsf{D}(\mathcal{A}) \simeq \mathrm{QCoh}([\mathrm{Spec}(\mathcal{A})/\widetilde{\mathbb{G}}_{m,R,\bullet}]).$$

11.2 Relation to the Hopf algebras of Ayoub

In [3] Ayoub develops a general framework (in the world of 1-categories) to construct bialgebra and Hopf algebra objets starting from data which involves a symmetric monoidal left adjoint $f\colon \mathcal{M} \to \mathcal{E}$ (and also a symmetric monoidal section of f) satisfying certain properties. If g denotes the right adjoint of f, it turns out $f(g(\mathbf{1}))$ is a bialgebra, and if another condition is satisfied, even a Hopf algebra in $\mathcal{E}$.

Applied to our case we have the following comparison statement:

Proposition 11.5. *Let R be a commutative ring and $a\colon \mathcal{A} \to R$ an augmented commutative algebra in* $\mathsf{D}(R)^{\mathbb{Z}}$*. Then the canonical symmetric monoidal left adjoint* $\mathsf{D}(\mathcal{A}) \to \mathsf{D}(R)$ *satisfies Ayoub's conditions and the corresponding Hopf algebra in* $\mathrm{ho}\mathsf{D}(R)$ *is the image of* $C(a)'_{\bullet} \rtimes \widetilde{\mathbb{G}}_{m,R,\bullet} \in \mathrm{Gp}(\mathrm{Aff}_{\mathsf{D}(R)})$ *in* $\mathrm{ho}\mathsf{D}(R)$*.*

11.3 Beyond mixed Tate motives

In [3] Ayoub proves far reaching properties of the motivic Galois group which arises when his construction (see previous section) is applied to the Betti realization functor from geometric motives over a field with a given complex embedding to complexes.

In [1] a homotopical motivic Hopf algebra for the whole category of geometric motives is constructed (which is in particular an affine derived group scheme in our sense), it would be interesting to compare it to Iwanari's tannakization.

Cao describes in [8] the thick triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, k a field of characteristic 0, generated by the motive of an elliptic curve without complex multiplication over k, as modules over a cycle cdga graded over the representations of GL_2.

In a similar vein Iwanari analyzes in [19] subcategories generated by certain abelian varieties.

In [22] Artin motives are considered.

Mixed Artin-Tate motives are also considered in [34] and [31] [32].

Constructions of mixed motives are given in [16].

11.4 Higher Tannaka duality

Let us here just mention other works in the direction of higher Tannakian duality: Bhatt [4], Bhatt and Halpern-Leistner [5], Fukuyama-Iwanari [12], Iwanari [20] Lurie [28], Pridham [33], Toën [38] [39], Wallbridge [42].

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