Harris's Theorem and its applications to some kinetic and biological modes

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Outline

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   - Some history
   - Harris's Theorem
2. Linear kinetic equations
   - Linear BGK, Linear Boltzmann equations
   - Convergence results
3. Structured population dynamics
   - Elapsed-time structured neuron population models
   - Size structured population models

- Wolfgang (Vincent) Doeblin (1915-1940)
- P.G. Bergman, J.L. Lebowitz 1955: Convergence to equilibrium for scattering equations with non-equilibrium steady states by using Doeblin's Theorem (non-quantitative)
- T. E. Harris 1956: Conditions for existence and uniqueness of a steady state for a Markov process
- S. P. Meyn, R. L. Tweedie 1993: Exponential convergence to equilibrium
- J.C. Mattingly, A. M. Stuart, D.J. Higham 2001: Convergence to equilibrium for the kinetic Fokker-Planck equation (non-quantitative)

- M. Hairer, J. Mattingly 2011: Simplified proof using mass transport distances,
  Quantitative rates for convergence to equilibrium once assumptions verified quantitatively
- E.A. Carlen, R. Esposito, J.L. Lebowitz, R. Marra, C. Mouhot 2016: Exponential convergence to a non-equilibrium steady state for some non-linear kinetic equations on the torus by using Doeblin's Theorem (quantitative)

- $(\Omega, \mathcal{F})$: measurable space with Borel $\sigma-$algrebra,
- $\mathcal{M}(\Omega)$: space of finite measures on $(\Omega, \mathcal{F})$,
- $\mathcal{P}(\Omega)$: space of probability measures on $(\Omega, \mathcal{F})$.
- Markov process x on a state space $\Omega \approx$ *transition probability functions*
- $S : \Omega \times \mathcal{S} \mapsto \mathbb{R}$ is a *transition probability function* on a finite measure space if
  1. $S(x, \cdot)$ is a probability measure for every $x \in \Omega$,
  2. $x \mapsto S(x, A)$ is a measurable function for every $A \in \mathcal{S}$.

- Stochastic/ Markov operator on probability measures $P : \Omega \to \mathcal{P}(\Omega)$ acting on
  1. the space of finite measures on Ω:
  $$(P\mu)(A) = \int_\Omega S(x, A)\mu(\mathrm{d}x),$$
  2. the space of bounded measurable functions $\varphi : \Omega \to [0, \infty)$:
  $$(P\varphi)(x) = \int_\Omega \varphi(y)S(x, \mathrm{d}y).$$
- continuous time Markov processes $\approx$ a family of Markov transition kernels / semigroup
- $P_t : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$, *linear, mass* and *positivity preserving*, satisfying
  1. the semigroup property: $P_{s+t} = P_s P_t$, for all $t, s \geq 0$.
  2. P_0 is the identity.
- $P_t\mu$ is the weak solution to the PDE with initial data μ.

Hypothesis 1: Doeblin's condition

We assume that $(P_t)_{t\geq 0}$ is a stochastic semigroup, defined through a Markov transition probability function, and that there exists $t_0 > 0$, a probability distribution ν and $\alpha \in (0,1)$ such that for any x in the state space Ω we have

$$P_{t_0}\delta_x \geq \alpha\nu. \tag{1}$$

Doeblin's Theorem

If we have a stochastic semigroup $(P_t)_{t\geq 0}$ satisfying Doeblin's condition then for any two measures μ_1 and μ_2 and any integer $n \geq 0$ we have that

$$\|P_{t_0}^n \mu_1 - P_{t_0}^n \mu_2\|_{\mathrm{TV}} \leq (1-\alpha)^n \|\mu_1 - \mu_2\|_{\mathrm{TV}}. \qquad (2)$$

As a consequence, the semigroup has a unique equilibrium probability measure μ_*, and for all μ

$$\|P_t(\mu - \mu_*)\|_{\mathrm{TV}} \leq \frac{1}{1-\alpha} e^{-\lambda t} \|\mu - \mu_*\|_{\mathrm{TV}}, \qquad t \geq 0, \qquad (3)$$

where

$$\lambda := \frac{\log(1-\alpha)}{t_0} > 0.$$

Hypothesis 2: Lyapunov condition

There exists some function $V : \Omega \to [0, \infty)$ and constants $D \geq 0, \gamma \in (0,1)$ such that

$$P_{t_0}(V)(x) \leq \gamma V(x) + D. \tag{4}$$

- This is equivalent to the statement with $\gamma = e^{-\lambda t_0}$ and $D = \frac{K}{\lambda}(1 - e^{-\lambda t_0}) \leq K t_0$:

$$\int_\Omega f(t_0, x) V(x) \mathrm{d}x \leq \gamma \int_\Omega f(0, x) V(x) \mathrm{d}x + D. \tag{5}$$

- $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(t,x) V(x) \mathrm{d}x \leq -\lambda \int_\Omega f(t,x) V(x) \mathrm{d}x + K.$

Hypothesis : 'local' Doeblin-like condition

There exists a probability measure ν and a constant $\alpha \in (0,1)$ such that

$$\inf_{x\in\mathcal{C}} P_{t_0}\delta_x \geq \alpha\nu, \tag{6}$$

where

$$\mathcal{C} = \{x : V(x) \leq R\}$$

for some $R > \frac{2D}{1-\gamma}$.

- Distance on probability measures for every $\beta > 0$ defined as

$$\rho_\beta(\mu_1, \mu_2) = \int (1 + \beta V(x)) |\mu_1 - \mu_2| (\mathrm{d}x).$$

- A weighted supremum norm for every measurable function φ for every $\beta > 0$ as in

$$\|\varphi(x)\| = \sup_x \frac{|\varphi(x)|}{1 + \beta V(x)}.$$

Harris's Theorem

If Hypotheses 4 and 6 hold then there exist $\bar{\alpha} \in (0,1)$ and $\beta > 0$ such that

$$\rho_\beta(\mathcal{P}_{t_0}\mu_1, \mathcal{P}_{t_0}\mu_2) \leq \bar{\alpha}\rho_\beta(\mu_1, \mu_2). \tag{7}$$

- Explicitly if we choose $\epsilon \in (0, \alpha)$ and $\delta \in \left(\gamma + \frac{2D}{R}, 1\right)$, then we can set $\beta = \frac{\epsilon}{D}$ and $\bar{\alpha} = \max\left\{1 - \alpha + \epsilon, \frac{2 + R\beta\delta}{2 + R\beta}\right\}$.

Subgeometric Harris's Theorem [Hairer '16]

Given the forwards operator, $\mathcal{L}$ of the stochastic semigroup P_t s.t. $\mathcal{L}\phi := \frac{\mathrm{d}}{\mathrm{d}t} S_t \phi\big|_{t=0}$,
suppose that there exists a continuous function V valued in $[1, \infty)$ with pre compact level sets such that

$$\mathcal{L}V \leq K - \phi(V),$$

for some constant K and some strictly concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(0) = 0$ and increasing to infinity.
Assume that for every $C > 0$ we have the minorisation condition: for some time t_0, a probability distribution ν and $\alpha \in (0, 1)$, then for all x with $V(x) \leq C$

$$P_{t_0} \delta_x \geq \alpha \nu.$$

Subgeometric Harris's Theorem [Hairer '16]

With these conditions we have

- There exists a unique invariant measure μ for the Markov process and it satisfies

$$\int \phi(V(x))\mathrm{d}\mu \leq D.$$

- Let H_ϕ be the function defined by

$$H_\phi = \int_1^u \frac{\mathrm{d}s}{\phi(s)}$$

then there exists a constant C such that for all ν

$$\|P_t\nu - \mu\|_{\mathrm{TV}} \leq \frac{C\nu(V)}{H_\phi^{-1}(t)} + \frac{C}{(\phi \circ H_\phi^{-1})(t)}.$$

Hypocoercivity of linear kinetic equations via Harris's Theorem

joint work with **José A. Cañizo** (U. Granada), **Chuqi Cao** (Paris-Dauphine) and **Josephine Evans** (Paris-Dauphine)

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f, \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d,$$
$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L} f, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

where

- $f = f(t, x, v)$ with time $t \geq 0$,
- $\mathcal{L}$ (generator of a stochastic semigroup) acts only on v,
  - linear relaxation Boltzmann (linear BGK) operator
  - linear Boltzmann operator
- Φ is a confining potential.

We consider the linear relaxation Boltzmann equation,

- in $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$:

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}f = \mathcal{L}^+ f - f,$$
$$\mathcal{L}^+ f = \left(\int f(t, x, u) \mathrm{d}u \right) \mathcal{M}(v), \quad \mathcal{M}(v) := (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}}. \tag{8}$$

- in $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ with periodic B.C.:

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - f, \tag{9}$$

- M.J. Cáceres, J.A. Carrillo, T. Goudon 2003: Convergence to equilibrium in H^1 at a rate faster than any function of t,
- C. Mouhot, L. Neuman 2006; F. Hérau 2006; J. Dolbeault, C. Mouhot, C. Schmeiser 2015: Convergence exponentially fast in both H^1 and L^2 using hypocoercivity techniques.

We consider the linear Boltzmann equation,

- in $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$:

$$\partial_t f + v\cdot\nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = Q(f, \mathcal{M}), \quad \mathcal{M}(v) := (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}},$$
$$Q(f, g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, \sigma)(f(v')g(v'_*) - f(v)g(v_*))\mathrm{d}\sigma\mathrm{d}v_*,$$
$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad (10)$$

- in $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ with periodic B.C.:

$$\partial_t f + v \cdot \nabla_x f = Q(f, \mathcal{M}), \quad (11)$$

- Q is the *Boltzmann operator*
- B is the *collision kernel*, assumed to be hard and s.t.

$$B(|v - v_*|, \sigma) = |v - v_*|^\gamma \; b\left(\sigma \cdot \frac{v - v_*}{|v - v_*|}\right), \qquad (12)$$

  for some $\gamma \geq 0$.
- b integrable in σ, uniformly positive on $[-1, 1]$; i.e. there exists $C_b > 0$ s.t.

$$b(z) \geq C_b \text{ for all } z \in [-1, 1] \qquad (13)$$

- B. Lods, C. Mouhot, G. Toscani 2008; M. Bisi, J.A. Cañizo, B. Lods 2015; J.A. Cañizo, A. Einav, B. Lods 2017: Spatially homogeneous case
- C. Mouhot, L. Neuman 2006; J. Dolbeault, C. Mouhot, C. Schmeiser 2015: Convergence exponentially fast in both H^1 and L^2 using hypocoercivity techniques.

Theorem (Cañizo, Cao, Evans & Y. '19): on the torus

Suppose that $t \mapsto f_t$ is the solution to either linear BGK or linear Boltzmann equation on the torus with initial data $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$.
In the case of linear Boltzmann equation we also assume (12) with $\gamma \geq 0$, and (13). Then there exist constants $C > 0$, $\lambda > 0$ (independent of f_0) such that

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*, \tag{14}$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (that is, $\mu(x,v) = \mathcal{M}(v)$). The norm is the total variation norm $\|\cdot\|_{\mathrm{TV}}$,

$$\|f_0 - \mu\|_* = \|f_0 - \mu\|_{\mathrm{TV}} := \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |f_0 - \mu| \mathrm{d}x\mathrm{d}v \text{ for equation (9),}$$

and it is a weigthed total variation norm,

$$\|f_0 - \mu\|_* = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} (1 + |v|^2)|f_0 - \mu| \mathrm{d}x\mathrm{d}v \text{ for equation (11).}$$

Idea of the proof:

- $t \mapsto T_t f_0$ solves the equation $\partial_t f + v \cdot \nabla_x f = 0$ with initial condition f_0.
- In this case: $T_t f_0(x, v) = f_0(x - tv, v)$.
- By Duhamel's formula

$$e^t f_t \geq \int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \mathrm{d}r \mathrm{d}s.$$

- Bound on the 'jump' operator:
Lemma $\mathcal{L}$: For all $\delta_L > 0$ there exists $\alpha_L > 0$ s.t. for all nonnegative functions $g \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ we have

$$\mathcal{L}^+ g(x, v) \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x, u) \mathrm{d}u \right) 1_{\{|v| \leq \delta_L\}},$$

and for almost all $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$.

Idea of the proof:

- Bound on the 'transport' part:
  Lemma T: Given any time $t_0 > 0$ and radius $R > 0$ there exists $\delta_L, R' > 0$ s.t. for all $t \geq t_0$ it holds that

$$\int_{B(R')} T_t \left(\delta_{x_0} 1_{\{|v| \leq \delta_L\}}\right) \mathrm{d}v \geq \frac{1}{t^d} 1_{\{|x| \leq R\}},$$

  for all x_0 with $|x_0| < R$.
- For the linear Boltzmann: suppose $\gamma \geq 0$ st.

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - \sigma(v) f,$$

  where $\sigma(v) \geq 0$ and $\sigma(v)$ behaves like $|v|^\gamma$ for large v, i.e.

$$0 \leq \sigma(v) \leq (1 + |v|^2)^{\gamma/2} \text{ for } v \in \mathbb{R}^d.$$

Theorem (Cañizo, Cao, Evans & Y. 19'): on $\mathbb{R}^d$

Suppose that $t \mapsto f_t$ is the solution to either linear BGK or linear Boltzmann equation in the whole space with initial data $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and with a confining potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ bounded below s.t. for some positive constants γ_1, γ_2, A:

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d.$$

In the case of linear Boltzmann equation we also assume (12), (13) and for some positive constants γ_1, γ_2, A:

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d.$$

Theorem (Cañizo, Cao, Evans & Y. '19): on $\mathbb{R}^d$

Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that

$$\|f_t - \mu\|_* \le Ce^{-\lambda t}\|f_0 - \mu\|_* \tag{15}$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\mathrm{d}\mu = \mathcal{M}(v)e^{-\Phi(x)}\mathrm{d}v\mathrm{d}x.$$

The norm $\|\cdot\|_*$ is a weighted total variation norm defined by

$$\|f_t - \mu\|_* := \int \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) |f_t - \mu|\mathrm{d}v\mathrm{d}x.$$

Idea of the proof:

- linear relaxation Boltzmann case:
  - Minorisation condition: instantaneously producing large velocities under the action of ϕ.
  - *Lyapunov condition* is satisfied for:

$$V(x,v) = 1 + \Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4}x \cdot v + \frac{1}{8}|x|^2$$

    under the given assumptions on Φ.
- linear Boltzmann case:
  - Similar arguments for minorisation condition
  - Lyapunov condition: only Maxwell molecules case ($\gamma = 0$) with the same functional above.

Theorem (Cañizo, Cao, Evans & Y. '19): subgeometric

Suppose that $t \mapsto f_t$ is the solution to the linear BGK equation in $\mathbb{R}^d$ with a confining potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$. Assume that for some β in $(0,1)$, Φ satisfies for some positive constants γ_1, γ_2, A:

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

Then we have that there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min \Bigg\{ \|f_0 - \mu\|, \\ C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2 \right) (1+t)^{-\beta/(1-\beta)} \Bigg\}. \quad (16)$$

Theorem (Cañizo, Cao, Evans & Y. '19): subgeometric

Suppose that $t \mapsto f_t$ is the solution to the linear Boltzmann equation in $\mathbb{R}^d$, satisfies (12), (13) and for some positive constants $\gamma_1, \gamma_2, A, \beta, \gamma_3$:

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\beta+1} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \leq \gamma_3 \langle x \rangle^{1+\beta},$$

Then we have that there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min \Big\{ \|f_0 - \mu\|, \\ C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|\right)(1+t)^{-\beta} \Big\}.$$

- D. Bakry, P. Cattiaux, A. Guillin 2008; R. Douc, G. Fort, A. Guillin 2009: C. Cao 2018: Subgeometric convergence for kinetic Fokker-Planck equations with weak confinement

Remarks:

We obtain via Harris's Theorem

- exponential convergence rates
  - on the $d-$dimesional torus
  - in the whole space with a confining potentials growing at least quadratically at ∞.
- algebraic convergence rates for subquadratic potentials
  - this is the only work showing this type of convergence in a quantitative way for the equations we present.
- in TV norms or weighted TV norms, (alternatively L^1 or weighted L^1 norms)
- for much wider range of initial conditions,
  - for initial conditions with slow decaying tails,
  - for measure initial conditions with very bad local regularity.
- existence of stationary solutions under quite general conditions

Asymptotic behaviour of neuron population models structured by elapsed-time

joint work with **José A. Cañizo** (U. Granada)

- P. Gabriel 2017: Exponential convergence to equilibrium for the conservative renewal equation
- G. Dumont, P. Gabriel 2017: Exponential convergence to equilibrium for leaky integrate-and-fire neuron model
- V. Bansaye, B. Cloez, P.Gabriel 2017: Quantitative estimates for some non-conservative and non-homogeneous positive semigroups, new bounds on the homogeneous setting
- V. Bansaye, B. Cloez, P.Gabriel, A. Marguet 2019: Non-conservative semigroups, quantitative estimates based on a non-homogenous h-transform of the semigroup and the construction of Lyapunov functions.

1. Age-structured neuron population model

$$\begin{aligned} \frac{\partial}{\partial t} n(t,s) + \frac{\partial}{\partial s} n(t,s) + p(N(t),s)n(t,s) = 0, \quad & t,s \geq 0, \\ N(t) := n(t,0) = \int_0^{+\infty} p(N(t),s)n(t,s)\mathrm{d}s, \quad & t > 0, \\ n(0,s) = n_0(s) \geq 0, \quad & s > 0. \end{aligned} \tag{17}$$

- $n(t,s)$: a population density function giving the probability of finding a neuron in state s at time t.
- *s: time elapsed since the last discharge.
- $p(N,s)$: firing rate of neurons in the the state s, in an environment N resulting from the global neural activity.
- $N(t)$: density of neurons having a discharge at time t.

1. Age-structured neuron population model

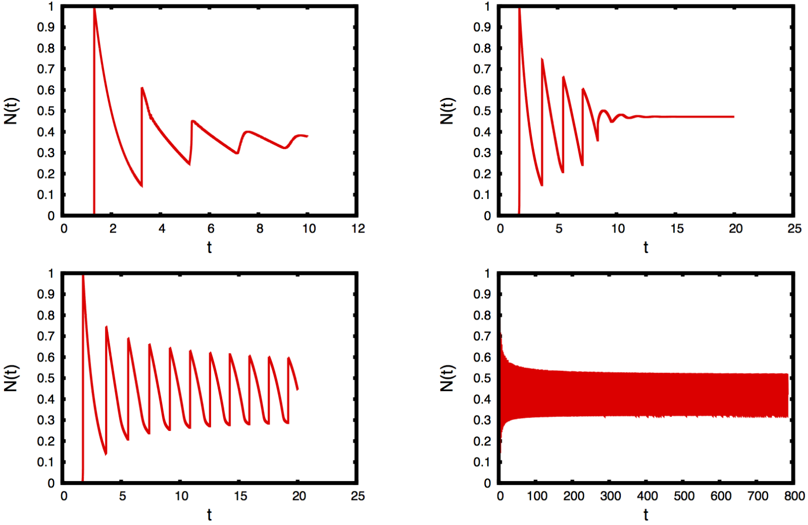
Proposed in Pakdaman, Perthame & Salort 2010:

$$\frac{\partial}{\partial t}n(t,s) + \frac{\partial}{\partial s}n(t,s) + p(X(t),s)n(t,s) = 0, \quad t,s \geq 0,$$
$$N(t) := n(t,0) = \int_0^{+\infty} p(X(t),s)n(t,s)\mathrm{d}s, \quad t > 0,$$
$$n(0,s) = n_0(s) \geq 0, \quad s > 0.$$

where

$$X(t) = J\int_0^u \alpha(u)N(t-u)\mathrm{d}u.$$

1. Age-structured neuron population model



2. Neuron population model with fatigue

Proposed in Pakdaman, Perthame & Salort 2014:

$$\begin{aligned} \frac{\partial}{\partial t} n(t,s) + \frac{\partial}{\partial s} n(t,s) + p(N(t),s)n(t,s) \\ = \int_0^{+\infty} \kappa(s,u) p(N(t),u) n(t,u) \mathrm{d}u, \quad t,s \geq 0, \\ n(t,s=0) = 0, \quad N(t) = \int_0^{+\infty} p(N(t),s) n(t,s) \mathrm{d}s, \\ n(t=0,s) = n_0(s) \geq 0, \quad s > 0. \end{aligned} \tag{18}$$

where $n(t,s)$, $p(N,s)$ and $N(t)$ same as before and $\kappa(s,u) \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^+)$: distribution of neurons which take the state s when a discharge occurs after an elapsed time u since their last discharge.
$\kappa(s,u) = \delta_0(s)$ recovers the first model.

Properties

- mass conservative: $\frac{d}{dt}\int_0^{+\infty} n(t,s)ds = 0$.
- positivity preserving.
- $n_0(s) \in \mathcal{P}$, $\int_0^{+\infty} n_0(s)ds = 1$.
- larger the stimulation neurons induce smaller the refractory period.
  i.e. excitatory network: $\frac{\partial}{\partial N} p(N,s) > 0$.
- $\kappa(s,u) = \delta_0(s)$ in (18) we recover the first model.

Assumptions

We make the following assumptions for models (17) and (18)(last one is only for (18));

1. $p \in W^{1,\infty}([0,+\infty) \times [0,+\infty)),$ s.t. $p(N,s) \geq 0 \quad \forall N, s \geq 0,$ with $\mathbf{L}$ being the smallest number s.t. $|p(N_1,s) - p(N_2,s)| \leq \mathbf{L}|N_1 - N_2|$ for $N_1, N_2 \geq 0$ and $s > 0$.
2. $\exists s_*, p_{\min}, p_{\max} > 0, \quad \forall s \geq 0,\ p_{\min} \mathbb{1}_{[s_*,\infty)} \leq p(\cdot, s) \leq p_{\max}.$
3. $\frac{\partial}{\partial s} p(N,s) > 0$, for all $N, s \geq 0$.
4. For each $u \geq 0$, $\kappa(\cdot, u) \in \mathcal{P}(\mathbb{R}^+)$ supported on $[0,u]$ and $\exists \epsilon > 0, 0 < \delta < s_*$ s.t. $\kappa(\cdot, u) \geq \epsilon \mathbb{1}_{[0,\delta]}$ for all $u \geq s_*$, $\int_0^u \kappa(s,u)ds = 1$.

Theorem (Cañizo & Y. '18)

Suppose that (1)–(3) are satisfied for equation (17), or (1)–(4) for equation (18). Suppose also that L is small enough depending on p and κ. Let n_0 be a probability measure on $[0, +\infty)$.
Then, there exists a unique probability measure n_* which is a stationary solution to (17) or (18), and there exist constants $C \geq 1$, $\lambda > 0$ depending only on p and κ such that the (mild or weak) measure solution $n = n(t)$ to (17)-(18) satisfies

$$\|n(t) - n_*\|_{\mathrm{TV}} \leq Ce^{-\lambda t}\|n_0 - n_*\|_{\mathrm{TV}}, \text{ for all } t \geq 0. \tag{19}$$

Remark:

Constants are constructive.
To be precise one can take

$$\lambda = \lambda_1 - \tilde{C}, \quad C = C_1 \text{ for } (17),$$
$$\lambda = \lambda_2 - \tilde{C}, \quad C = C_2 \text{ for } (18),$$

with $\beta = p_{\min} e^{-2p_{\max} s_*}$ and $\tilde{C} = 2p_{\max} \frac{L}{1-L}$, where

$$C_1 := \frac{1}{1 - s_* \beta}, \qquad \lambda_1 = -\frac{\log(1 - s_* \beta)}{2s_*}$$
$$C_2 := \frac{1}{1 - \epsilon\delta(s_* - \delta)\beta}, \qquad \lambda_2 = -\frac{\log(1 - \epsilon\delta(s_* - \delta)\beta)}{2s_*}$$

Remark:

Smallness condition on L can be written as

$$L < \min\left\{\frac{p_{\min}^2}{p_{\max}^2\left(s_* p_{\min}(s_* p_{\min}+2)+2\right)}, \frac{\log(1-s_*\beta)}{\log(1-s_*\beta)-4p_{\max}s_*}\right\},$$

for (17) or

$$L < \min\left\{\frac{p_{\min}\epsilon\delta(s_*-\delta)\beta}{p_{\min}\epsilon\delta(s_*-\delta)\beta+p_{\max}e^{4p_{\max}s_*}}, \frac{\log(1-\epsilon\delta(s_*-\delta))}{\log(1-\epsilon\delta(s_*-\delta))-4p_{\max}s_*}\right\},$$

for (18).

Idea of the proof:

1. Positive lower bound for solutions of the linear equation
2. Positive lower bound 1 $\implies$ Doeblin condition satisfied
3. Doeblin condition $\implies$ spectral gap
4. Pertubation argument applied to the linear equation $\implies$ exponential relaxation to the stationary solution

Idea of the proof for the main theorem:

- Define two operators;

$$\mathcal{L}_{N(t)} n := \partial_t n = -\partial_s n - p(N(t), s) n + \int \kappa(s,u) p(N(t), u) n \mathrm{d}u.$$

$$\mathcal{L}_{N_*} n := -\partial_s n - p(N_*, s) n + \int \kappa(s,u) p(N_*, u) n \mathrm{d}u.$$

Idea of the proof for the main theorem:

- Define two operators;

$$\mathcal{L}_{N(t)}n := \partial_t n = -\partial_s n - p(N(t),s)n + \int \kappa(s,u)p(N(t),u)n\mathrm{d}u.$$

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Idea of the proof for the main theorem:

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Idea of the proof for the main theorem:

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- $n(t,s) - n_* = S_t n_0(s) - n_* + \int_0^t S_{t-\tau} h(\tau,s) \mathrm{d}\tau$
- $\|n(t) - n_*\|_{\mathrm{TV}} \leq \|S_t n_0 - n_*\|_{\mathrm{TV}} + \left\| \int_0^t S_{t-\tau} h(\tau,s) \mathrm{d}\tau \right\|_{\mathrm{TV}}.$
- "h" lemma: $\|h(t)\|_{\mathrm{TV}} \leq \tilde{C} \|n(t) - n_*\|_{\mathrm{TV}}$ where $\tilde{C}$ calculated explicitly.
- Result by Grönwall's argument: $\|n(t) - n_*\|_{\mathrm{TV}} \leq C e^{-(\lambda - \tilde{C})t} \|n_0 - n_*\|_{\mathrm{TV}}.$

Proof of the "h" lemma:

$$
\begin{aligned}
\|h(t)\|_{\mathrm{TV}} = \|(\mathcal{L}_{N_*} - \mathcal{L}_{N(t)})n(t,s)\|_{\mathrm{TV}} \\
\leq \|(p(N(t),s) - p(N_*,s))n(t,s)\|_{\mathrm{TV}} + \\
\left\| \int_0^{+\infty} \kappa(s,u)(p(N_*,u) - p(N(t),u))n(t,u)\mathrm{d}u \right\|_{TV} \\
\leq L\|n(t)\|_{\mathrm{TV}}|N_* - N(t)| + L\|n(t)\|_{\mathrm{TV}}|N_* - N(t)| \\
\leq 2p_{\max}\frac{L\|n(t)\|_{\mathrm{TV}}}{1 - L\|n(t)\|_{\mathrm{TV}}}\|n(t) - n_*\|_{\mathrm{TV}} \\
= \underbrace{\frac{2p_{\max}L}{1-L}}_{\tilde{C}}\|n(t) - n_*\|_{\mathrm{TV}}
\end{aligned}
$$

Relaxation to equilibrium for the growth-fragmentation equation by Harris's Theorem

joint work with **José A. Cañizo** (U. Granada) and **Pierre Gabriel** (U. Versailles)

The growth-fragmentation equation

$$\frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}(g(x)n(t,x)) = \int_x^\infty \kappa(y,x)n(t,y)\mathrm{d}y - B(x)n(t,x), \quad t,x \geq 0,$$

coupled with $n(t,0) = 0, t > 0$ and $n(0,x) = n_0(x), x > 0$.

- cell division, polymerisation, neurosciences, prion proliferation, telecommunication (TCP, IP), ecology...
- unicellular organisms: age ??, elapsed-time ??, mass of the cell ✓, length of the cell ✓, DNA content ✓, level of certain proteins ✓
- $B(x) = 1 \implies$ mass is conserved.

The growth-fragmentation equation

- $n(t,x)$: the population density of individuals structured by a variable $x > 0$ at a time $t \geq 0$.
- $g\ (0,+\infty) \to (0,+\infty)$ is the *growth rate*.
- B: *total division/fragmentation rate* of individuals with size $x \geq 0$. $\to B(x) = \int_0^y \frac{y}{x}\kappa(x,y)\mathrm{d}y$.
- $\kappa(y,x)$: the rate at which individuals of size x are obtained as the result of a fragmentation event of an individual of size y.
  1. equal mitosis: $\kappa(x,y) = B(x)\frac{2}{x}\delta_{\{y=\frac{x}{2}\}}$.

     $$\partial_t n(t,x) + \partial_x(g(x)n(t,x) + B(x)n(t,x) = 4B(2x)n(t,2x).$$

  2. uniform fragmentation: $\kappa(x,y) = B(x)\frac{2}{x}$

Perron eigenvalue problem:

Finding suitable eigenelements $(\lambda, N(x), \phi(x))$ which satisfy:

$$\begin{aligned} &\frac{\partial}{\partial x}(g(x)N(x)) + (B(x)+\lambda)N(x) = \int_x^{+\infty} \kappa(x,y)N(y)\mathrm{d}y, \\ &g(0)N(0) = 0, \quad N(x) \geq 0, \quad \int_0^{+\infty} N(x)\mathrm{d}x = 1. \end{aligned} \tag{20}$$

$$\begin{aligned} &-g(x)\frac{\partial}{\partial x}\phi(x) + (B(x)+\lambda)\phi(x) = \int_0^{x} \kappa(y,x)\phi(y)\mathrm{d}y, \\ &\phi(x) \geq 0, \quad \int_0^{+\infty} \phi(x)N(x)\mathrm{d}x = 1. \end{aligned} \tag{21}$$

Scaled equation

- scaling $\rightsquigarrow m(t,x) := n(t,x)e^{-\lambda t}$:

$$\frac{\partial}{\partial t} m(t,x) + \frac{\partial}{\partial x}(g(x)m(t,x)) = \int_x^\infty \kappa(y,x)m(t,y)\mathrm{d}y - (B(x)+\lambda)m(t,x), \quad t,x \geq 0,$$
$$m(t,0) = 0, \quad t > 0, \quad m(0,x) = n_0(x), \quad x > 0. \quad (22)$$

- conserved quantity: $\frac{\mathrm{d}}{\mathrm{d}t}\int \phi(x)m(t,x)\mathrm{d}x = 0.$

Assumptions

- $g\ (0,+\infty) \to (0,+\infty)$ is a locally Lipschitz.
  There exists $C > 0$ such that $g(x) \leq Cx$ for all $x \geq 1$.
  $\int_0^1 \frac{1}{g(x)} \mathrm{d}x < +\infty$.
- $B\ (0,+\infty) \to (0,+\infty)$ s.t. $B(x) \underset{x \to +\infty}{\longrightarrow} +\infty$.
- Example: $g(x) = x^\alpha$ where $\alpha \in [0,1]$ and $B(x) = x^\gamma$, where $\gamma > 0$.
- When $g(x) = x$: $B(x) \underset{x \to 0}{\longrightarrow} 0$
- [E. Bernard, M. Doumic & P. Gabriel 2019] mitosis with $g(x) = x \rightsquigarrow \lambda_k = 1 + \frac{2ik\pi}{\log 2}, k \in \mathbb{Z}$

Thank you!