Two-fluid Navier-Stokes-Maxwell system with Ohm's law limit from two-species Vlasov-Maxwell-Boltzmann system

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Two-species Vlasov-Maxwell-Boltzmann system

$$\begin{cases} \partial_t F^+ + v\cdot\nabla_x F^+ + \frac{q^+}{m^+}(E + v\times B)\cdot\nabla_v F^+ = Q(F^+,F^+) + Q(F^+,F^-) \\ \text{(Vlasov-Boltzmann equation for cations)} \\ \partial_t F^- + v\cdot\nabla_x F^- - \frac{q^-}{m^-}(E + v\times B)\cdot\nabla_v F^- = Q(F^-,F^-) + Q(F^-,F^+)\,, \\ \text{(Vlasov-Boltzmann equation for anions)} \\ \mu_0\varepsilon_0\partial_t E - \nabla_x\times B = -\mu_0\int_{\mathbb{R}^3}(q^+F^+ - q^-F^-)v\,\mathrm{d}v\,, \quad \text{(Ampere)} \\ \partial_t B + \nabla_x\times E = 0\,, \quad \text{(Faraday)} \\ \mathrm{div}_x E = \frac{1}{\varepsilon_0}\int_{\mathbb{R}^3}(q^+F^+ - q^-F^-)\,\mathrm{d}v\,, \quad \text{(Gauss)} \\ \mathrm{div}_x B = 0\,. \quad \text{(Gauss)} \end{cases}$$

- Evolution of a gas of two species of oppositely charged particles (cations of charge $q^+ > 0$ and mass $m^+ > 0$, and anions of charge $-q^- < 0$ and mass $m^- > 0$), subject to auto-induced electromagnetic forces.
- $F^\pm(t,x,v) \geq 0$: particle number density, distribution of the positively (negatively) charged ions, i.e. cations (anions).

Two-species Vlasov-Maxwell-Boltzmann system

- The evolution of the densities $F^{\pm}$ are governed by the Vlasov-Boltzmann equation. It tells that variation of the densities $F^{\pm}$ along the trajectories of the particles are subject to the influence of a Lorentz force and inter-particel collisions in the gas.
- The Lorentz force acting on the gas is auto-induced. That is, the electric field $E(t, x)$ and the magnetic field $B(t, x)$ are generated by the motion of the particles in the plasma itself.
- Their motion is governed by the Maxwell's equation, which are the remaining equations, namely Ampère equation, Faraday's equation and Gauss' laws.
- The physical constants $\mu_0, \varepsilon_0 > 0$ are the vacuum permeability (or magnetic constant) and the vacuum permittivity (or electric constant). Note that their relation to the speed of light is the formula $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

Boltzmann collision kernel

$$Q(f,h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}(f'h'_* - fh_*)b(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\,,$$

where $\omega \in \mathbb{S}^2$ and

$$b(v-v_*,\omega) = |v-v_*|^\gamma \hat{b}(\cos\theta)$$

for $\gamma \in [0,1]$. For convenience, we take $\hat{b}(\cos\theta)$ such that $\int_{\mathbb{S}^2}\hat{b}(\cos\theta)\mathrm{d}\omega = 1$. Here we have used the standard abbreviations

$$f = f(v)\,,\ f' = f(v')\,,\ h_* = h(v_*)\,,\ h'_* = h(v'_*)$$

with (v', v'_*) given by

$$\begin{aligned} v' &= v + [(v-v_*)\cdot\omega]\omega\,,\\ v'_* &= v_* - [(v-v_*)\cdot\omega]\omega\,, \end{aligned}$$

which denotes velocities after a collision of particles having velocities v, v_* before the collision.

Global-in-time well-posedness of VMB

- Renormalized solutions.
  - DiPerna-Lions (cutoff kernels, Boltzmann, Vlasov-Poisson-Boltzmann, late 80's);
  - Alexandre-Villani (non-cutoff Boltzmann, 2000);
  - Arsenio-Saint-Raymond (VMB, 2016): very difficult, hyperbolic feature.
  - Advantage: ”large” initial data, only require some physical bounds (momentum, energy, entropy...)
  - Drawbacks: renormalization, uniqueness not known, conservation laws with defect measures...
- Classical solutions near equilibriums (Maxwellian).
  - Ukai (cutoff kernels, 70's)
  - Guo (cutoff kernels, 2003);
  - Gressman-Strain, Alexandre-Morimoto-Ukai-Xu-Yang (non-cutoff kernels, 2010);
  - Many other people made several generalizations.
  - Advantage: smooth solutions.
  - Drawbacks: near global Maxwellian, not a ”large” solution.

Dissipation Properties

Boltzmann's *H*-Theorem (1872): the dissipation of the entropy and equilibrium.

$$\langle \log(f)\mathcal{B}(f,f)\rangle = -\iiint \tfrac{1}{4}\log\left(\frac{f_1' f'}{f_1 f}\right)(f_1' f' - f_1 f)\, b\, \mathrm{d}\omega\, \mathrm{d}v_1\, \mathrm{d}v$$

$$\leq 0 \quad \text{for “every”} \quad f = f(v)\,.$$

Moreover, for “every” $f = f(v)$ the following are equivalent:

- $\langle \log(f)\mathcal{B}(f,f)\rangle = 0$,
- $\mathcal{B}(f,f) = 0$,
- f is a Maxwellian, which have the form

$$f = \mathcal{M}(v;\rho,u,\theta) = \frac{\rho}{(2\pi\theta)^{\frac{D}{2}}}\exp\left(-\frac{|v-u|^2}{2\theta}\right),$$

where (ρ, u, θ) can be recovered from the Maxwellian by

$$\rho = \langle f\rangle\,, \quad \rho u = \langle v f\rangle\,, \quad \rho\theta = \tfrac{1}{D}\langle |v-u|^2 f\rangle\,.$$

Nondimensionalization

- Length scale λ_o:

$$\int_\Omega \mathrm{d}x = \lambda_o^D \, .$$

- Density and velocity scales ρ_o, $\theta_o^{1/2}$:

$$\iint_\Omega F^{in} \mathrm{d}v\mathrm{d}x = \rho_o \lambda_o^D \, , \quad \iint_\Omega \tfrac{1}{2}|v|^2 F^{in} \mathrm{d}v\mathrm{d}x = \tfrac{D}{2} \rho_o \theta_o \lambda_o^D \, .$$

- Nondimensional velocity, space, and time:

$$v = \sqrt{\theta_o}\,\hat{v} \, , \quad x = \lambda_o \hat{x} \, , \quad t = \frac{\lambda_o}{\sqrt{\theta_o}} \hat{t} \, .$$

- The equilibrium associated with the initial data F^{in}:

$$M_o \equiv \mathcal{M}(\rho_o \, , 0 \, , \theta_o) = \frac{\rho_o}{(2\pi\theta_o)^{D/2}} \exp\left(-\frac{|v|^2}{2\theta_o} \right) .$$

Knudsen Number, Rescaled Boltzmann Equation

- the collision operator determines a time scale τ_o by

$$\iiint M_{o1} M_o b(\omega, v_1 - v) \mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v = \frac{\rho_o}{\tau_o} \,.$$

  This time is on the order of the time interval that molecules in the equilibrium density M_o spend traveling freely between collisions, the so-called mean free time.
- The length scale of the mean free path $= \tau_o \times \sqrt{\theta_o}$.
- Nondimensional kinetic density:

$$F(t, x, v) = \frac{\rho_o}{\theta_o^{D/2}} \hat{F}(\hat{t}, \hat{x}, \hat{v}) \,,$$

- Nondimensinal collision operator:

$$\mathcal{B}(F, F)(t, x, v) = \frac{\rho_o}{\tau_o \theta_o^{D/2}} \hat{\mathcal{B}}(\hat{F}, \hat{F})(\hat{t}, \hat{x}, \hat{v}) \,.$$

Knudsen Number, Rescaled Boltzmann Equation

- After the nondimensionalization, the rescaled Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\varepsilon}\mathcal{B}(F,F)\,, \quad F(t,x,v) = F^{in}(x,v)\,,$$

- Knudsen number:

$$\varepsilon = \frac{\theta_o^{1/2}\tau_o}{\lambda_o} = \frac{\text{mean-free-path}}{\text{macroscopic length scale}}\,,$$

- Fluid Dynamics Regimes: ε very small.
- The nondimensional equilibrium associated to this problem is the so-called global Maxwellian

$$M \equiv \mathcal{M}(1,0,1) = \frac{1}{(2\pi)^{D/2}} \exp\left(-\frac{|v|^2}{2}\right).$$

Fluid Dynamics

We consider fluid dynamical regimes in which F is close to the global Maxwellian $M = M(v)$.

- Fluctuations:

$$F_\varepsilon(t,x,v) = M\left[1 + \delta_\varepsilon g_\varepsilon(t,x,v)\right] \,,$$

- $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Physically, δ_ε is the Mach number.
- Fluctuations equation:

$$\partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon = \frac{\delta_\varepsilon}{\varepsilon} \mathcal{Q}(g_\varepsilon, g_\varepsilon) \,.$$

- The linearized collision operator $\mathcal{L}$:

$$\mathcal{L}\tilde{g} = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (\tilde{g} + \tilde{g}_1 - \tilde{g}' - \tilde{g}_1') b(\omega, v_1 - v) \, \mathrm{d}\omega M_1 \, \mathrm{d}v_1 \,.$$

- Properties of $\mathcal{L}$:
  - Nonnegative, Self-adjoint.
  - $\mathrm{Null}(\mathcal{L}) = \mathrm{span}\{1, v_1, \cdots, v_D, |v|^2\}$.
- The fluctuations g_ε satisfy

$$\varepsilon\Big(\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon\Big) + \mathcal{L}g_\varepsilon = \delta_\varepsilon Q(g_\varepsilon, g_\varepsilon)\,.$$

- Formally let $\varepsilon \to 0$, $g_\varepsilon \to g$, then $\mathcal{L}g = 0$.
- g has the form: $g = \rho + v\cdot u + \left(\frac{1}{2}|v|^2 - \frac{D}{2}\right)\theta$.
- This form is call an infinitesimal Maxwellian. It comes from

$$\begin{aligned}\mathcal{M}(1+\delta\rho, \delta u, 1+\delta\theta) &= \frac{1+\delta\rho}{(2\pi(1+\delta\theta))^{D/2}}\exp\left(\frac{|v-\delta u|^2}{2(1+\delta\theta)}\right)\\ &= M\Big(1+\delta\, g + O(\delta^2)\Big)\,.\end{aligned}$$

From Boltzmann to incompressible Navier-Stokes

- Scaled Boltzmann equation:

$$\varepsilon\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon)\,,\quad F_\varepsilon(t,x,v) = F_\varepsilon^{in}(x,v)\,,$$

- $F_\varepsilon = M(1+\varepsilon g_\varepsilon)$ and $F_\varepsilon^{in} = M(1+\varepsilon g^{in})$;
- $g_\varepsilon \to -\theta(t,x) + v\cdot u(t,x) + \theta(t,x)(\frac{|v|^2}{2} - \frac{3}{2})$, and (u,θ) satisfy incompressible Navier-Stokes-Fourier system:

$$\begin{aligned} \partial_t u + u\cdot\nabla_x u + \nabla_x p &= \mu\Delta_x u\,,\quad u(x,0) = u^0(x)\\ \nabla_x\cdot u &= 0\,,\\ \partial_t\theta + u\cdot\nabla_x\theta &= \kappa\Delta_x\theta\,,\quad \theta(x,0) = \theta^0(x)\,, \end{aligned}$$

  with initial data $u(0,x) = \langle v g^{in}\rangle$ and $\theta(0,x) = \langle(\frac{|v|^2}{5} - 1)g^{in}\rangle$, and μ and κ also could be determined.
- Rigorously justify this limit in the context of DiPerna-Lions solutions and classical solutions.

Hydrodynamic limits of Boltzmann equations

- Renormalized solutions
  - Bardos-Golse-Levermore (Navier-Stokes limit, 1994)
  - Golse-Levermore (Stokes & acoustic limit, 2002)
  - Saint-Raymond (Euler limit, 2003)
  - Golse-Saint-Raymond (Navier-Stokes limit, 2004, 2009)
  - Masmoudi-Saint-Raymond, J-Masmoudi (Navier-Stokes limit in bounded domain, 2003, 2015)
- Classical solutions
  - Caflisch (Compressible Euler limit, 1980)
  - Bardos-Ukai (Navier-Stokes limit, 1991, semigroup method)
  - Guo (Navier-Stokes limit, 2006, nonlinear energy method)
  - J-Xu-Zhao (Navier-Stokes limit, 2014)
  - Briant, Briant-Merino-Mouhot (Navier-Stokes limit, 2015, 2018)
  - Gallagher-Tristani (Navier-Stokes limit limit, 2019)
- Question: How about hydrodynamic limits of VPB and VMB ? (VPB is similar to Boltzmann, but VMB is quite different and much harder! mainly because of its hyperbolic feature)

There are two types of results:

- Type I: From solutions of the kinetic equations, prove the compactness of the solution, then take limit in the conservation laws of the original kinetic equation, to obtain solutions of the fluid equations.
- Type II: From solutions of the fluid equations, construct solutions of the original kinetic equations with special form (usually in the expansion, for the Knudsen number ε very small.)
- Usually, type I results are harder to be obtained.
- Mouhot-Mischler in their “Kac’s program in kinetic theory”: This provides a first answer to the question raised by Kac. However, our answer is an “inverse” answer in the sense that our methodology is “*top-down*” from the limit equation to the many-particle system rather than “*bottom-up*” as was expected by Kac.

Hydrodynamic limits of VMB

- Jang-Masmoudi (2012): Formal analysis, both compressible and incompressible models, Navier-Stokes-Maxwell, Euler Maxwell, etc.
- Jang (2008): classical solutions, diffusive limit, but no magnetic effect.
- Arsenio-Saint-Raymond (2016): renormalized solutions, two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with Ohm's law. (huge book)
- J-Luo-Zhang (2018): classical solutions, same limit as Arsenio-Saint-Raymond's: using Hilbert expansion approach.
- J-Luo (2019): classical solutions, same limit as Arsenio-Saint-Raymond's.

Incompressible NSFM with Ohm's law

$$\begin{cases} \partial_t u + u \cdot \nabla_x u - \mu \Delta_x u + \nabla_x p = \frac{1}{2} nE + \frac{1}{2} j \times B \,, \\ \mathrm{div}_x u = 0 \,, \\ \partial_t E - \nabla_x \times B = -j \,, \\ \partial_t B + \nabla_x \times E = 0 \,, \\ j = nu + \sigma(-\frac{1}{2}\nabla_x n + E + u \times B) \,, \\ \mathrm{div}_x E = n \,,\ \mathrm{div}_x B = 0 \,, \end{cases} \tag{0.1}$$

- Global in time Leray type weak solutions are open, even in 2D.
- Masmoudi (2010): 2D case, global strong solutions for the initial data $L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$ with $s > 0$.
- Ibrahim-Keraani (2011): $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \times (\dot{H}^{1/2}(\mathbb{R}^3))^2$ for 3D, and $\dot{B}^0_{2,1}(\mathbb{R}^2) \times (L^2_{log}(\mathbb{R}^2))^2$ for 2D.
- Germain, Ibrahim and Masmoudi (2014).
- Arsenio-Ibrahim-Masmoudi: derivation of MHD from NSM.
- Arsenio-Gallagher (2018) : Weak solutions with finite energy, small electromagnetic field in $\dot{H}^s$ for $s \in [1/2, 3/2)$.

$$
\begin{cases}
\varepsilon\partial_t F_\varepsilon^\pm + v\cdot\nabla_x F_\varepsilon^\pm \pm (\varepsilon E_\varepsilon + v\times B_\varepsilon)\cdot\nabla_v F_\varepsilon^\pm = \frac{1}{\varepsilon}Q(F_\varepsilon^\pm, F_\varepsilon^\pm) + \frac{1}{\varepsilon}Q(F_\varepsilon^\pm, F_\varepsilon^\mp)\,, \\
F_\varepsilon^\pm = M(1+\varepsilon g_\varepsilon^\pm)\,, \\
\partial_t E_\varepsilon - \nabla_x \times B_\varepsilon = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3}(g_\varepsilon^+ - g_\varepsilon^-) vM \mathrm{d}v\,, \\
\partial_t B_\varepsilon + \nabla_x \times E_\varepsilon = 0\,, \\
\mathrm{div} E_\varepsilon = \int_{\mathbb{R}^3}(g_\varepsilon^+ - g_\varepsilon^-) M \mathrm{d}v\,, \\
\mathrm{div} B_\varepsilon = 0\,,
\end{cases}
$$

- Formally, $g_\varepsilon^\pm \to \rho^\pm(t,x) + u(t,x)\cdot v + \theta(t,x)(\frac{|v|^2}{2} - \frac{3}{2})$.
- $\rho = \frac{1}{2}(\rho^+ + \rho^-) = -\theta$ and $n = \rho^+ - \rho^-$. Then the functions (u, θ, n, E, B) obey the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with Ohm's law.
- Rigorously justify this limit.

Hilbert expansion-1

- Caflisch (1980, compressible Euler limit of Boltzmann equation before shock)
- Take the ansatz:

$$F_\varepsilon^\pm = M\Big\{1 + \varepsilon[g_0^\pm + \varepsilon\overline{g}_1^\pm + \varepsilon^2\overline{g}_2^\pm + \varepsilon g_{R,\varepsilon}^\pm]\Big\},$$
$$E_\varepsilon = E_0(t,x) + \varepsilon\overline{E}_1(t,x) + \varepsilon E_{R,\varepsilon}(t,x),$$
$$B_\varepsilon = B_0(t,x) + \varepsilon\overline{B}_1(t,x) + \varepsilon B_{R,\varepsilon}(t,x).$$

- the leading term $g_0^\pm$ is given by

$$g_0^\pm(t,x,v) = \rho_0^\pm(t,x) + u_0(t,x)\cdot v + \theta_0(t,x)\left(\frac{|v|^2}{2} - \frac{3}{2}\right), \quad (0.2)$$

and the functions $\overline{g}_i^\pm(t,x,v)$ $(i = 1,2)$ are of the form given later. We denote $\rho_0 = \frac{1}{2}(\rho_0^+ + \rho_0^-) = -\theta_0$ and $n_0 = \rho_0^+ - \rho_0^-$. Then the functions $(u_0, \theta_0, n_0, E_0, B_0)$ obey incompressible NSFM system with Ohm's law.

- the functions $\overline{E}_1(t,x)$, $\overline{B}_1(t,x)$ and $\overline{n}_1(t,x)$ appeared on the definitions of $\overline{g}_i^{\pm}(t,x,v)$, satisfies the linear Maxwell-type system, namely

$$\begin{cases} \partial_t \bar{E}_1 - \nabla_x \times \bar{B}_1 = -j_1\,, \\ \partial_t \bar{B}_1 + \nabla_x \times \bar{E}_1 = 0\,, \\ \mathrm{div}_x \bar{E}_1 = \bar{n}_1\,,\ \mathrm{div}_x \bar{B}_1 = 0\,, \\ j_1 = \bar{n}_1(u_0 \cdot \mathbf{M} + \theta_0 V) + u_1 \\ +\sigma(-\frac{1}{2}\nabla_x \bar{n}_1 + \bar{E}_1 + u_0 \times \bar{B}_1 + u_1 \times B_0) + \sum \Gamma_0^- U_{\Upsilon^-}\,. \end{cases}$$

Hilbert expansion-1

- the remainder terms $(g^{\pm}_{R,\varepsilon}, E_{R,\varepsilon}, B_{R,\varepsilon})$ subject to the system

$$\begin{cases} \varepsilon\partial_t G_{R,\varepsilon} + v\cdot\nabla_x G_{R,\varepsilon} + \mathcal{T}(v\times B_0)\cdot\nabla_v G_{R,\varepsilon} + \mathcal{T}(v\times B_{R,\varepsilon})\cdot\nabla_v G_0 \\ \qquad -E_{R,\varepsilon}\cdot v\mathcal{T}_1 + \frac{1}{\varepsilon}\mathbb{L}G_{R,\varepsilon} = \varepsilon H_{R,\varepsilon}\,, \\ \partial_t E_{R,\varepsilon} - \nabla_x\times B_{R,\varepsilon} = -\frac{1}{\varepsilon}\langle G_{R,\varepsilon}\cdot\mathcal{T}_1 v\rangle\,, \\ \partial_t B_{R,\varepsilon} + \nabla_x\times E_{R,\varepsilon} = 0\,, \\ \mathrm{div}_x E_{R,\varepsilon} = \langle G_{R,\varepsilon}\cdot\mathcal{T}_1\rangle\,,\ \mathrm{div}_x B_{R,\varepsilon} = 0\,, \end{cases}$$

Some notations

- for any $G = (g^+, g^-)^\top$,

$$\mathbb{L}G = \begin{pmatrix} \mathcal{L}g^+ + \mathcal{L}(g^+, g^-) \\ \mathcal{L}g^- + \mathcal{L}(g^+, g^-) \end{pmatrix},$$

where $\mathcal{L}g = -\frac{1}{M}\Big[Q(Mg, M) + Q(M, Mg)\Big]$, and
$\mathcal{L}(g, h) = -\frac{1}{M}\Big[Q(Mg, M) + Q(M, Mh)\Big]$.

- the projection operator $\mathbb{P}$ from L^2_v to $\ker(\mathbb{L})$ as

$$\mathbb{P}G = \rho^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u \cdot \begin{pmatrix} v \\ v \end{pmatrix} + \theta \begin{pmatrix} \frac{|v|^2}{2} - \frac{3}{2} \\ \frac{|v|^2}{2} - \frac{3}{2} \end{pmatrix},$$

where $\rho^\pm = \langle g^\pm \rangle$, $u = \Big\langle v \frac{g^+ + g^-}{2} \Big\rangle$ and $\theta = \Big\langle (\frac{|v|^2}{3} - 1) \frac{g^+ + g^-}{2} \Big\rangle$.

Instant energy

- energy functional

$$\widetilde{\mathbb{E}}_{N,l}(G,E,B) = \|E\|^2_{H_x^{N+1}} + \|B\|^2_{H_x^{N+1}} + \sum_{|m|\le N+1} \|\partial^m G\|^2_{L^2_{x,v}}$$
$$+ \sum_{|m|+|\beta|\le N} \|w^l \partial^m_\beta \mathbb{P}^\perp G\|^2_{L^2_{x,v}} + \sum_{\substack{|m|+|\beta|\le N+1 \\ \beta\neq 0}} \|w^l \partial^m_\beta \mathbb{P}^\perp G\|^2_{L^2_{x,v}}$$
$$+ \mathcal{E}_{0,N+5}(t) + \mathcal{E}_{1,N+3}(t)\,,$$

$$\mathcal{E}_{0,s}(t) = \|u_0\|^2_{H_x^s} + \|\theta_0\|^2_{H_x^s} + \tfrac{3}{2}\|E_0\|^2_{H_x^s} + \tfrac{5}{4}\|n_0\|^2_{H_x^s} + (\tfrac{3}{2} - \delta + \delta\sigma)\|B_0\|^2_{H_x^s}$$
$$+ (1-\delta)\|\partial_t B_0\|^2_{H_x^s} + \|\nabla_x B_0\|^2_{H_x^s} + \delta\|\partial_t B_0 + B_0\|^2_{H_x^s},$$

$$\mathcal{E}_{1,M}(t) = \|\bar{E}_1\|^2_{H_x^M} + \|\bar{n}_1\|^2_{H_x^M} + (1-\delta+\delta\sigma)\|\bar{B}_1\|^2_{H_x^M} + \|\nabla_x \bar{B}_1\|^2_{H_x^M}$$
$$+ (1-\delta)\|\partial_t \bar{B}_1\|^2_{H_x^M} + \delta\|\partial_t \bar{B}_1 + \bar{B}_1\|^2_{H_x^M}\,.$$

Dissipation rate

$$
\begin{aligned}
&\mathbb{D}_{N,l}(G,E,B)\\
&= \|E\|^2_{H^{N-1}_x} + \|\nabla_x B\|^2_{H^{N-1}_x} + \|\partial_t B\|^2_{H^{N-1}_x} + \mathcal{D}_{0,N+5}(t) + \mathcal{D}_{1,N+3}(t)\\
&+ \sum_{|m|\le N+1} \|\partial^m \mathbb{P} G\|^2_{L^2_{x,v}} + \frac{1}{\varepsilon^2} \sum_{\substack{|m|+|\beta|\le N+1\\ \beta\neq 0}} \|w^l \partial^m_\beta \mathbb{P}^\perp G\|^2_{L^2_{x,v}(\nu)}\\
&+ \frac{1}{\varepsilon^2} \sum_{|m|\le N+1} \|\partial^m \mathbb{P}^\perp G\|^2_{L^2_{x,v}(\nu)} + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta|\le N} \|w^l \partial^m_\beta \mathbb{P}^\perp G\|^2_{L^2_{x,v}(\nu)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{0,s}(t) &= \mu\|\nabla_x u_0\|^2_{H^s_x} + \tfrac{\kappa}{2}\|\theta_0\|^2_{H^s_x} + \sigma\|E_0\|^2_{H^s_x} + \tfrac{3}{2}\sigma\|n_0\|^2_{H^s_x} + \tfrac{1}{2}\sigma\|\nabla_x n_0\|^2_{H^s_x}\\
&+(\sigma-\delta)\|\partial_t B_0\|^2_{H^s_x} + \delta\|\nabla_x B_0\|^2_{H^s_x}\\
&+ \tfrac{1}{2}\sigma \sum_{m\le s} \| -\tfrac{1}{2}\nabla_x \partial^m n_0 + \partial^m E_0 + (\partial^m u_0)\times B_0\|^2_{L^2_x}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{1,M}(t) &= \tfrac{\sigma}{2}\|\bar{E}_1\|^2_{H^M_x} + \tfrac{3}{4}\sigma\|\bar{n}_1\|^2_{H^M_x} + \tfrac{1}{4}\sigma\|\nabla_x \bar{n}_1\|^2_{H^M_x} + \tfrac{\delta}{2}\|\nabla_x \bar{B}_1\|^2_{H^M_x}\\
&+ \tfrac{(\sigma-\delta)}{2}\|\partial_t \bar{B}_1\|^2_{H^M_x} + \tfrac{1}{2}\|\nabla_x \bar{u}_1\|^2_{H^M_x}.
\end{aligned}
$$

Classical solution for NSFM (J-Luo, 2017)

$(u_0^{in}, \theta_0^{in}, E_0^{in}, B_0^{in}) \in H_x^s \times H_x^s \times H_x^{s+1} \times H_x^{s+1}$, If there is $\lambda_0(s)$ depending only on s, μ, κ, and σ, s.t. $\mathcal{E}_{0,s}^{in} \leq \lambda_0(s)$. Then the NSFM admits a unique global-in-time solution $(u_0^{in}, \theta_0^{in}, E_0^{in}, B_0^{in})$ satisfying

$$
\begin{aligned}
& u_0, \theta_0 \in L^\infty(\mathbb{R}^+, H_x^s) \cap L^2(\mathbb{R}^+, \dot{H}_x^{s+1}) \\
& E_0 \in L^\infty(\mathbb{R}^+, H_x^s), \quad n_0 (= \mathrm{div}_x E_0) \in L^\infty(\mathbb{R}^+, H_x^s) \cap L^2(\mathbb{R}^+, \dot{H}_x^{s+1}) \\
& B_0 \in L^\infty(\mathbb{R}^+, H_x^{s+1}), \quad \partial_t B_0 (= -\nabla_x \times B_0) \in L^\infty(\mathbb{R}^+, H_x^s).
\end{aligned} \tag{0.3}
$$

Moreover, for any $t \geq 0$, there holds the following energy inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{0,s}(t) + \mathcal{D}_{0,s}(t) \leq 0, \tag{0.4}
$$

and consequently, there exists some constant $C = C(\mu, \kappa, \sigma) > 0$, such that

$$
\begin{aligned}
& \sup_{t \geq 0} (\|u_0\|_{H_x^s}^2 + \|\theta_0\|_{H_x^s}^2 + \|E_0\|_{H_x^s}^2 + \|n_0\|_{H_x^s}^2 + \|B_0\|_{H_x^s}^2 + \|\partial_t B_0\|_{H_x^s}^2) \\
& \quad + \int_0^\infty (\mu \|\nabla_x u_0\|_{H_x^s}^2 + \kappa \|\nabla_x \theta_0\|_{H_x^s}^2 + \sigma \|\nabla_x n_0\|_{H_x^s}^2) \mathrm{d}t \leq C \mathcal{E}_{0,s}^{in}.
\end{aligned} \tag{0.5}
$$

Main Theorem (J-Luo-Zhang, 2018)

Let $N \geq 4$ and $l \geq 2\gamma + 1$. Given u_0^{in}, θ_0^{in}, E_0^{in}, B_0^{in}, $\overline{E}_1^{in}$, $\overline{B}_1^{in}$, $E_{R,\varepsilon}^{in}$, $B_{R,\varepsilon}^{in}$ and $G_{R,\varepsilon}^{in}$, let the initial data of VMB "well-prepared". If there are two small constants ε_0, $\eta_0 > 0$, depending only on μ, σ, κ, l and N, s.t. $\varepsilon \in (0, \varepsilon_0)$ and

$$\widetilde{\mathbb{E}}_{N,l}(G_{R,\varepsilon}^{in}, E_{R,\varepsilon}^{in}, B_{R,\varepsilon}^{in}) \leq \eta_0 \,,$$

then VMB admits a global-in-time classical solution $(F_\varepsilon^\pm, E_\varepsilon, B_\varepsilon)$ belonging to $L^\infty(\mathbb{R}^+; H_{x,v}^{N+1} \times H_x^{N+1} \times H_x^{N+1})$ with special form, in which $(u_0, \theta_0, n_0, E_0, B_0) \in$ is a unique solution the Navier-Stokes-Fourier-Maxwell system with Ohm's law (0.1) with initial data $(u_0^{in}, \theta_0^{in}, \mathrm{div}_x E_0^{in}, E_0^{in}, B_0^{in})$.
Moreover,

$$\begin{aligned}
&\sup_{t \geq 0} \mathbb{E}_{N,l}(G_{R,\varepsilon}, E_{R,\varepsilon}, B_{R,\varepsilon})(t) \leq \mathbb{E}_{N,l}(G_{R,\varepsilon}^{in}, E_{R,\varepsilon}^{in}, B_{R,\varepsilon}^{in})(0) \\
&\leq C\widetilde{\mathbb{E}}_{N,l}(G_{R,\varepsilon}^{in}, E_{R,\varepsilon}^{in}, B_{R,\varepsilon}^{in}) \,.
\end{aligned}$$

holds for some constant $C > 0$, depending only on μ, σ, κ, l and N.

Energy estimate for Maxwell-type system

Let the initial data $(\bar{E}_1^{in}, \bar{B}_1^{in}) \in H_x^{M+1} \cap H_x^{M+1}$ with $M \geq 1$. Assume that there exists some small constant $\lambda_1 = \lambda_1(M, \mu, \kappa, \sigma) \in (0, \lambda_0]$, such that $\mathcal{E}_{0,M+1}^{in} \leq \lambda_1$. Then smooth solution $(\bar{\rho}_1, \bar{u}_1, \bar{\theta}_1, \bar{E}_1, \bar{B}_1)$ to the linear Maxwell-type equation obey the following bounds

$$\|\bar{u}_1\|^2_{H_x^{M+1}}(t) \leq C(1 + \mathcal{E}_{0,M+1}(t))\mathcal{D}_{0,M+1}(t),$$

$$\|\bar{\rho}_1\|^2_{H_x^{M+1}}(t) = \|\bar{\theta}_1\|^2_{H_x^{M+1}}(t) \leq C\,\mathcal{E}_{0,M+1}(t)\Big(1 + \mathcal{E}_{0,M+1}(t)\Big)\mathcal{D}_{0,M+1}(t)$$

and $$\frac{\mathrm{d}}{\mathrm{dt}}[\mathcal{E}_{1,M}(t) + \widetilde{C}_M \mathcal{E}_{0,M+2}(t)] + [\mathcal{D}_{1,M}(t) + \mathcal{D}_{0,M+2}(t)] \leq 0,$$

with the constants $C = C(\mu, \kappa, \sigma) > 0$ and $\widetilde{C}_M = \widetilde{C}_M(M, \mu, \kappa, \sigma) > 1$. Moreover, for any $t \geq 0$, the following energy bounds holds:

$$\|\bar{E}_1\|^2_{H_x^M}(t) + \|\mathrm{div}_x \bar{E}_1\|^2_{H_x^M}(t) + \|\nabla_x \times \bar{E}_1\|^2_{H_x^M}(t) + \|\bar{B}_1\|^2_{H_x^M}(t)$$
$$\leq C\Big(\mathcal{E}_{0,M+2}(t) + \mathcal{E}_{1,M}(t)\Big).$$

Uniform spacial estimate for remainder system

$$\begin{cases} \varepsilon\partial_t G_R + v\cdot\nabla_x G_R + \mathcal{T}(v\times B_0)\cdot\nabla_v G_R + \mathcal{T}(v\times B_R)\cdot\nabla_v G_0 \\ \qquad\qquad -E_R\cdot v\mathcal{T}_1 + \frac{1}{\varepsilon}\mathbb{L}G_R = \varepsilon H_R\,, \\ \partial_t E_R - \nabla_x\times B_R = -\frac{1}{\varepsilon}\langle G_R\cdot\mathcal{T}_1 v\rangle\,, \\ \partial_t B_R + \nabla_x\times E_R = 0\,, \\ \mathrm{div}_x E_R = \langle G_R\cdot\mathcal{T}_1\rangle\,,\ \mathrm{div}_x B_R = 0\,. \end{cases}$$

1) We first estimate the hydrodynamic part $\mathbb{P}G_R$ of G_R,

$$\mathbb{P}G_R = \rho_R^+\begin{pmatrix}1\\0\end{pmatrix} + \rho_R^-\begin{pmatrix}0\\1\end{pmatrix} + u_R\cdot\begin{pmatrix}v\\v\end{pmatrix} + \theta_R\begin{pmatrix}\frac{|v|^2}{2}-\frac{3}{2}\\ \frac{|v|^2}{2}-\frac{3}{2}\end{pmatrix}$$

Our goal is to estimate $\rho_R^\pm(t,x)$, $u_R(t,x)$ and $\theta_R(t,x)$ in terms of $\mathbb{P}^\perp G_R$. 2) We then estimate the electric field E_R and the magnetic field B_R by finding some decay effects of E_R and B_R, which will play an essential role in establishing the global energy estimates.

Uniform spacial estimate for remainder system

There is a constant $C_1 > 0$ such that

$$\begin{aligned}
&\|u_R\|^2_{H^{N+1}_x} + \|\theta_R\|^2_{H^{N+1}_x} + \|\rho^+_R\|^2_{H^{N+1}_x} + \|\rho^-_R\|^2_{H^{N+1}_x} + \|\mathrm{div}_x E_R\|^2_{H^N_x} \\
&\leq C_1 \varepsilon \tfrac{\mathrm{d}}{\mathrm{d}t} \widetilde{\mathcal{A}}_N(t) + \frac{C_1}{\varepsilon^2} \sum_{|m|\leq N+1} \|\partial^m \mathbb{P}^\perp G_R\|^2_{L^2_{x,v}} \\
&+ C_1 \varepsilon^2 \sum_{|m|\leq N} \|\mathcal{P}_{\mathfrak{B}} \partial^m H_R\|^2_{L^2_{x,v}} + C_1 \mathcal{E}^{in}_{0,N+2} \|B_R\|^2_{H^N_x} \\
&+ C_1 \Big[\Big(\int_{\mathbb{T}} u_R \mathrm{d}x \Big)^2 + \Big(\int_{\mathbb{T}} \theta_R \mathrm{d}x \Big)^2 + \Big(\int_{\mathbb{T}} \rho^+_R \mathrm{d}x \Big)^2 + \Big(\int_{\mathbb{T}} \rho^-_R \mathrm{d}x \Big)^2 \Big]
\end{aligned} \tag{0.6}$$

for ε sufficiently small.

Uniform spacial estimate for remainder system

We remark that the last three terms of last estimate require to be estimated.

- The term $C_1\varepsilon^2 \sum_{|m|\le N} ||\mathcal{P}_{\mathfrak{B}}\partial^m H_R||^2_{L^2_{x,v}}$ will be controlled in estimating the mixed derivatives.
- The term $C_1\mathcal{E}^{in}_{0,N+2}||B_R||^2_{H^N_x}$ can be dominated by finding enough decay structures of the Maxwell equations on E_R, B_R in the remainder system.
- By analyzing the conservation laws of mass, momentum and energy of the remainder system , one can give an estimation on the last term including the integral forms.

Decay of B_R

Now we estimate the term $C_1\mathcal{E}^{in}_{0,N+2}||B_R||^2_{H^N_x}$. The *key point* is to find enough dissipation on B_R by making use of the Maxwell equations,

$$\begin{cases} \partial_t E_R - \nabla_x \times B_R = -\frac{1}{\varepsilon}\langle G_R \cdot \mathcal{T}_1 v\rangle = -\frac{1}{\varepsilon}\langle \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 v\rangle\,, \\ \partial_t B_R + \nabla_x \times E_R = 0\,, \\ \mathrm{div}_x E_R = \rho_R^+ - \rho_R^-\,,\ \mathrm{div}_x B_R = 0\,. \end{cases} \tag{0.7}$$

We can get $\partial_{tt} B_R - \Delta_x B_R = \frac{1}{\varepsilon}\nabla \times \langle \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 v\rangle$, whose dissipation is not enough. We try to derive the Ohm's law from the microscopic equation of G_R, which will supply a decay term $\partial_t B_R$.

$$\partial_{tt} B_R - \Delta_x B_R + \sigma \partial_t B_R = \nabla_x \times \mathcal{K}(\mathbb{P}^\perp G_R)\,, \tag{0.8}$$

Let $F_\varepsilon^\pm(t,x,v) = M(v) + \varepsilon\sqrt{M(v)}G_\varepsilon^\pm(t,x,v)$.This leads to the perturbed two-species Vlasov-Maxwell-Boltzmann system

$$\begin{cases} \partial_t G_\varepsilon + \frac{1}{\varepsilon}\Big[v\cdot\nabla_x G_\varepsilon + \mathrm{q}(\varepsilon E_\varepsilon + v\times B_\varepsilon)\cdot\nabla_v\Big]G_\varepsilon + \frac{1}{\varepsilon^2}LG_\varepsilon - \frac{1}{\varepsilon}(E_\varepsilon\cdot v)\sqrt{M}\mathrm{q}_1 \\ \qquad\qquad = \frac{1}{2}\mathrm{q}(E_\varepsilon\cdot v)G_\varepsilon + \frac{1}{\varepsilon}\Gamma(G_\varepsilon, G_\varepsilon)\,, \\ \partial_t E_\varepsilon - \nabla_x\times B_\varepsilon = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3} G_\varepsilon\cdot\mathrm{q}_1 v\sqrt{M}\mathrm{d}v\,, \\ \partial_t B_\varepsilon + \nabla_x\times E_\varepsilon = 0\,, \\ \mathrm{div}_x E_\varepsilon = \int_{\mathbb{R}^3} G_\varepsilon\cdot\mathrm{q}_1\sqrt{M}\mathrm{d}v\,,\ \mathrm{div}_x B_\varepsilon = 0\,, \end{cases} \tag{0.9}$$

To state our main theorems, we introduce the following energy functional and dissipation rate functional respectively

$$\mathbb{E}_s(G,E,B) = \|G\|^2_{H^s_{x,v}} + \|E\|^2_{H^s_x} + \|B\|^2_{H^s_x}\,,$$

$$\mathbb{D}_s(G,E,B) = \frac{1}{\varepsilon^2}\|\mathbb{P}^\perp G\|^2_{H^s_{x,v}(\nu)} + \|\nabla_x\mathbb{P}G\|^2_{H^{s-1}_x L^2_v} + \|E\|^2_{H^{s-1}_x} + \|\nabla_x B\|^2_{H^{s-2}_x}\,.$$

From VMB to NSM: uniform estimate (J-Luo 2019)

Theorem

For the integer $s \geq 3$ and $0 < \varepsilon \leq 1$, there are constants $\ell_0 > 0$, $c_0 > 0$ and $c_1 > 0$, independent of ε, such that if $\mathbb{E}_s(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in}) \leq \ell_0$, then the Cauchy problem (0.9) admits a global solution

$$G_\varepsilon(t,x,v) \in L_t^\infty(\mathbb{R}^+; H_{x,v}^s)\,, \mathbb{P}^\perp G_\varepsilon(t,x,v) \in L_t^2(\mathbb{R}^+; H_{x,v}^s(\nu))\,,$$
$$E_\varepsilon(t,x), B_\varepsilon(t,x) \in L_t^\infty(\mathbb{R}^+; H_x^s)$$

with the global uniform energy estimate

$$\sup_{t\geq 0} \mathbb{E}_s(G_\varepsilon, E_\varepsilon, B_\varepsilon)(t) + c_0 \int_0^\infty \mathbb{D}_s(G_\varepsilon, E_\varepsilon, B_\varepsilon)(t)\mathrm{d}t \leq c_1 \mathbb{E}_s(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in})\,. \tag{0.10}$$

From VMB to NSM: convergence

Theorem 2 (J-Luo 2019): Let $0 < \varepsilon \leq 1$, $s \geq 3$ and $\ell_0 > 0$ be as in Theorem 1. Assume that the initial data $(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in})$ satisfy

1. $G_\varepsilon^{in} \in H^s_{x,v}$, E_ε^{in}, $B_\varepsilon^{in} \in H^s_x$;
2. $\mathbb{E}_s(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in}) \leq \ell_0$;
3. there exist scalar functions $\rho^{in}(x)$, $\theta^{in}(x)$, $n^{in}(x) \in H^s_x$ and vector-valued functions $u^{in}(x)$, $E^{in}(x)$, $B^{in}(x) \in H^s_x$ such that

$$\begin{aligned} G_\varepsilon^{in} &\to G^{in} \quad \text{strongly in } H^s_{x,v}\,, \\ E_\varepsilon^{in} &\to E^{in} \quad \text{strongly in } H^s_x\,, \\ B_\varepsilon^{in} &\to B^{in} \quad \text{strongly in } H^s_x \end{aligned} \tag{0.11}$$

as $\varepsilon \to 0$, where $G^{in}(x, v)$ is of the form

$$\begin{aligned} G^{in}(x,v) =& (\rho^{in} + \tfrac{1}{2} n^{in}) \tfrac{\mathsf{q}_1 + \mathsf{q}_2}{2} \sqrt{M} + (\rho^{in} - \tfrac{1}{2} n^{in}) \tfrac{\mathsf{q}_2 - \mathsf{q}_1}{2} \sqrt{M} \\ & + u^{in} \cdot v \mathsf{q}_2 \sqrt{M} + \theta^{in} (\tfrac{|v|^2}{2} - \tfrac{3}{2}) \mathsf{q}_2 \sqrt{M}\,. \end{aligned}$$

From VMB to NSM: convergence (J-Luo 2019)

Let $(G_\varepsilon, E_\varepsilon, B_\varepsilon)$ be the family of solutions to VMB constructed in Theorem 1. Then, as $\varepsilon \to 0$,

$$G_\varepsilon / \sqrt{M} \to (\rho + \tfrac{1}{2}n)\frac{\mathsf{q}_1+\mathsf{q}_2}{2} + (\rho - \tfrac{1}{2}n)\frac{\mathsf{q}_2-\mathsf{q}_1}{2} + u \cdot v\mathsf{q}_2 + \theta(\tfrac{|v|^2}{2} - \tfrac{3}{2})$$

weakly-$\star$ in $t \geq 0$, strongly in $H^{s-1}_{x,v}$ and weakly in $H^s_{x,v}$, and

$$E_\varepsilon \to E \quad \text{and} \quad B_\varepsilon \to B$$

strongly in $C(\mathbb{R}^+; H^{s-1}_x)$, weakly-$\star$ in $t \geq 0$ and weakly in H^s_x. Here

$$(u, \theta, n, E, B) \in C(\mathbb{R}^+; H^{s-1}_x) \cap L^\infty(\mathbb{R}^+; H^s_x)$$

is the solution to the incompressible NSM with Ohm's law, with initial data

$$u|_{t=0} = \mathcal{P}u^{in}\,,\ \theta|_{t=0} = \tfrac{3}{5}\theta^{in} - \tfrac{2}{5}\rho^{in}\,,\ E|_{t=0} = E^{in}\,,\ B|_{t=0} = B^{in}\,,$$

Moreover, the convergence of the moments holds: as $\varepsilon \to 0$

$$\mathcal{P}\langle G_\varepsilon, \tfrac{1}{2}\mathsf{q}_2 v\sqrt{M}\rangle_{L^2_v} \to u\,, \quad \langle G_\varepsilon, \tfrac{1}{2}\mathsf{q}_2(\tfrac{|v|^2}{5} - 1)\sqrt{M}\rangle_{L^2_v} \to \theta\,,$$

strongly in $C(\mathbb{R}^+; H^{s-1}_x)$, weakly-$\star$ in $t \geq 0$ and weakly in H^s_x.

Conclusions and possible future works

- Two-fluid incompressible Navier-Stokes-Maxwell system can be justified from VMB.
- Extensions to more general kernels.
- Incompressible Euler-Maxwell system (already some very technical works on compressible E-M)
- Compressible Euler-Maxwell system.
- Compressible Navier-Stokes-Maxwell system (note: it is NOT similar to compressible Navier-Stokes!)
- Domains with boundary (very hard problem...)