TQFTs From Non-Semisimple Modular Categories and Modified Traces

Joint With A. Gainutdinov, N. Geer, B. Patureau, and I. Runkel

Marco De Renzi

Universität Zürich

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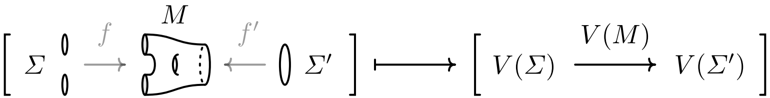
Topological Quantum Field Theories (TQFTs)

Definition ($(n+1)$-TQFT[1])

Symmetric monoidal functor from a category of $(n+1)$-cobordisms to the category of vector spaces

Schematically:

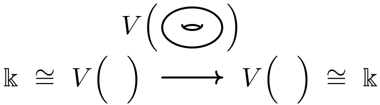
- closed n-manifold $\mapsto$ vector space (called *state space*)
- $(n+1)$-cobordism $\mapsto$ linear map
- gluing $\mapsto$ composition
- disjoint union $\mapsto$ tensor product



[1] Paraphrasing [A88]

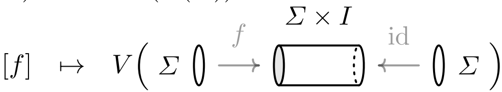
Why TQFTs?

- Topological invariants of closed $(n+1)$-manifolds



- Representations of mapping class groups of closed n-manifolds

$$\mathrm{Mod}(\Sigma) \to \mathrm{GL}(V(\Sigma))$$



From now on: $n = 2$

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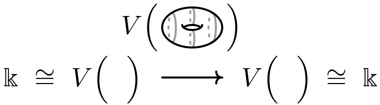

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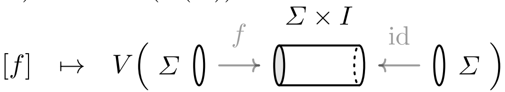

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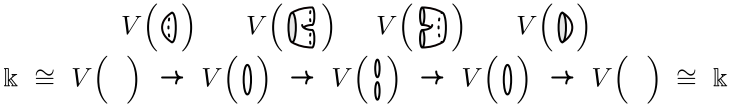

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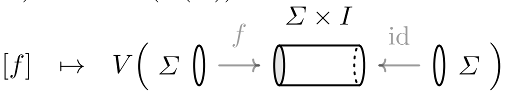


From now on: $n = 2$

Semisimple Theory

Turaev (1994)[2]

Modular (fusion) category $\bar{\mathscr{C}} \rightsquigarrow (2+1)$-TQFT $\mathrm{V}_{\bar{\mathscr{C}}}$

$\mathrm{V}_{\bar{\mathscr{C}}}$ induces invariant $\mathrm{RT}_{\bar{\mathscr{C}}}$ of 3-manifolds & projective representations $\bar{\rho}_{\mathrm{RT}}$ of mapping class groups of surfaces

Example: $\bar{\mathscr{C}} = \mathscr{C} / \mathscr{N}(\mathscr{C})$ for $\mathscr{C} = \bar{U}_q\mathfrak{g}\text{-mod}$ with q root of unity

Semisimple quotient of category of finite-dimensional representations of *small quantum group* $\bar{U}_q\mathfrak{g}$ with respect to *negligible morphisms*

Question:

- Can we avoid quotient operations?

[2] See [T94], building on [RT91], inspired by [W89]

Non-Semisimple Theory

Lyubashenko (1994)[3]

Modular category $\mathscr{C}$ $\rightsquigarrow$ invariant $\mathrm{L}_{\mathscr{C}}$ of 3-manifolds & projective representations $\bar{\rho}_{\mathrm{L}}$ of mapping class groups of surfaces

Example: $\mathscr{C} = \bar{U}_q\mathfrak{g}\text{-mod}$ with q root of unity

Category of finite-dimensional representations of small quantum group $\bar{U}_q\mathfrak{g}$

Questions:

- Does this construction carry more topological information?
- Can we extend it to TQFTs?

[3] See [L94], preceded by [H96], furthered in [KL01]

Problems

Ohtsuki (1995) + Kerler (1996)[4]

$\mathscr{C}$ non-semisimple, M closed 3-manifold, $b_1(M) > 0 \Rightarrow$

$$\mathrm{L}_{\mathscr{C}}(M) = 0$$

Obstruction for TQFTs

$\mathrm{V}_{\mathscr{C}}$ TQFT extending $\mathrm{L}_{\mathscr{C}} \Rightarrow \dim_{\Bbbk} \mathrm{V}_{\mathscr{C}}(\Sigma) = \mathrm{L}_{\mathscr{C}}(\Sigma \times S^1)$

Chen-Kuppum-Srinivasan (2007)[4]

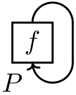
$\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$, $\bar{\mathscr{C}} = \mathscr{C}/\mathscr{N}(\mathscr{C}) \Rightarrow$

$$\mathrm{L}_{\mathscr{C}}(M) = h_1(M)\mathrm{RT}_{\bar{\mathscr{C}}}(M)$$

where $h_1(M) = |H_1(M)|$ if $b_1(M) = 0$, and $h_1(M) = 0$ otherwise

[4]See [O95, K96, CKS07]

Overcoming the Obstruction

$\mathscr{C}$ non-semisimple $\Rightarrow \operatorname{tr}_{\mathscr{C}}(f) =$  $= 0 \quad \begin{matrix} \forall\, P \in \operatorname{Proj}(\mathscr{C}) \\ \forall\, f \in \operatorname{End}_{\mathscr{C}}(P) \end{matrix}$

Renormalization

Replace $\operatorname{tr}_{\mathscr{C}}$ with non-degenerate modified trace t on $\operatorname{Proj}(\mathscr{C})$

- Used by[5]:
  - Costantino-Geer-Patureau
  - Murakami + Beliakova-Blanchet-Geer
  - D-Geer-Patureau
- Here:

  Lyubashenko invariant $\mathrm{L}_{\mathscr{C}} \rightsquigarrow$ renormalized invariant $\mathrm{L}'_{\mathscr{C}}$

[5] See [CGP12, M13, BBG17, DGP17]

Results

- $\mathscr{C}$ modular category over algebraically closed field $\Bbbk$
- $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$ category of admissible decorated cobordisms

Theorem (D-Gainutdinov-Geer-Patureau-Runkel[6])

There exists a $(2+1)$-TQFT

$$\mathrm{V}_{\mathscr{C}} : \check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}} \to \mathrm{Vect}_{\Bbbk}$$

extending both the renormalized invariant $\mathrm{L}'_{\mathscr{C}}$ and the projective representations $\bar{\rho}_{\mathrm{L}}$ of Lyubashenko

Roughly speaking:

- the topological information carried by $\mathrm{V}_{\mathscr{C}}$ is not contained in the one carried by $\mathrm{V}_{\bar{\mathscr{C}}}$ for $\bar{\mathscr{C}} = \mathscr{C}/\mathscr{N}(\mathscr{C})$

[6] See [DGGPR19, DGGPR20]

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Finite Ribbon Categories

Convention: $\Bbbk$ algebraically closed field, $\mathscr{C}$ linear category over $\Bbbk$

- $\mathscr{C}$ *finite* if $\mathscr{C} \cong A\text{-mod}$ for some finite-dimensional algebra A
- $\mathscr{C}$ *ribbon* if equipped with
  - tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
  - tensor unit $\mathbb{1} \in \mathscr{C}$
  - duality morphisms $\overleftarrow{\mathrm{ev}}_X : X^* \otimes X \to \mathbb{1}$, $\overleftarrow{\mathrm{coev}}_X : \mathbb{1} \to X \otimes X^*$, $\overrightarrow{\mathrm{ev}}_X : X \otimes X^* \to \mathbb{1}$, $\overrightarrow{\mathrm{coev}}_X : \mathbb{1} \to X^* \otimes X \quad \forall\, X \in \mathscr{C}$
  - braiding morphisms $c_{X,Y} : X \otimes Y \to Y \otimes X \quad \forall\, X, Y \in \mathscr{C}$
  - twist morphisms $\vartheta_X : X \to X \quad \forall\, X \in \mathscr{C}$

  satisfying several axioms[7]

[7] See [EGNO15]

Graphical Representation of Morphisms

$\mathrm{id}_X =$ 

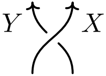
$\mathrm{id}_{X^*} =$ 

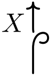
$\overleftarrow{\mathrm{ev}}_X =$ 

$\overleftarrow{\mathrm{coev}}_X =$ 

$\overrightarrow{\mathrm{ev}}_X =$ 

$\overrightarrow{\mathrm{coev}}_X =$ 

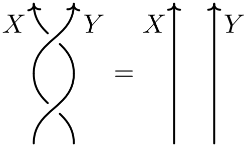
$c_{X,Y} =$ 

$\vartheta_X =$ 

- Composition: vertical stacking
- Tensor product: horizontal juxtaposition
- Axioms: invariance under isotopy & framed Reidemeister moves

Modular Categories

- $X \in \mathscr{C}$ *transparent* if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y} \quad \forall Y \in \mathscr{C}$



- $\mathscr{C}$ *factorizable* if $X \in \mathscr{C}$ transparent $\Rightarrow X \cong \bigoplus_{k=1}^{n} \mathbb{1}$

Definition (Modular Category[8])

Finite factorizable ribbon category

- $\mathscr{C}$ modular and semisimple $\Rightarrow \mathscr{C}$ modular in the sense of Turaev

From now on: $\mathscr{C}$ modular category

[8] See [S16] for equivalence with definition of [L94]

Example: Hopf Algebra Structure on $\bar{U}_q\mathfrak{sl}_2$

- $q = e^{\frac{2\pi i}{r}}$ with $3 \leqslant r \in \mathbb{Z}$ odd
- $\bar{U}_q\mathfrak{sl}_2$ complex algebra with[9]
  - generators E, F, K
  - relations $E^r = F^r = 0, \quad K^r = 1,$
    $KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$
- Hopf algebra structure given by

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E & \varepsilon(E) &= 0 & S(E) &= -EK^{-1} \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1 & \varepsilon(F) &= 0 & S(F) &= -KF \\ \Delta(K) &= K \otimes K & \varepsilon(K) &= 1 & S(K) &= K^{-1} \end{aligned}$$

- $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$ category of finite-dimensional representations

[9] See [L90]

Example: Ribbon Structure on $\bar{U}_q\mathfrak{sl}_2$

- $\{a\} := q^a - q^{-a}, \quad [a] := \frac{\{a\}}{\{1\}}, \quad [a]! := [a][a-1]\cdots[1] \quad \forall\, a \in \mathbb{N}$
- R-matrix[10]

$$R := \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a$$

- Ribbon element

$$v := \frac{i^{\frac{r-1}{2}}}{\sqrt{r}} \sum_{a,b=0}^{r-1} \frac{\{-1\}^a}{[a]!} q^{-\frac{a(a-1)}{2} + \frac{(r+1)(a-b-1)^2}{2}} F^a K^b E^a$$

- $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$ is a modular category[10]

[10] See [L93, M95]

Coends

- $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$, $(X, Y) \mapsto X^* \otimes Y$
- *Dinatural transformation of source* H:
  - object $D \in \mathscr{C}$
  - family $\{d_X : X^* \otimes X \to D\}_{X \in \mathscr{C}}$ of morphisms satisfying

$$d_X \circ (f^* \otimes \mathrm{id}_X) = d_Y \circ (\mathrm{id}_{Y^*} \otimes f) \quad \forall\, X, Y \in \mathscr{C}, f \in \mathscr{C}(X, Y)$$

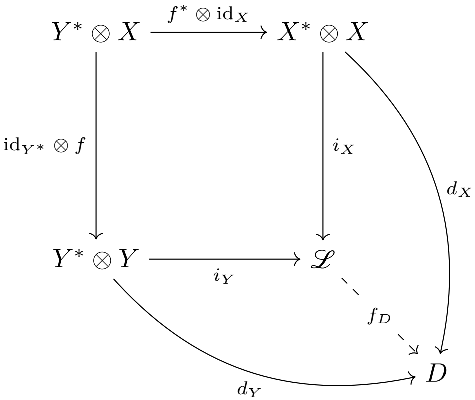
Definition (Coend (of H))

Universal dinatural transformation of source H, given by

- object $\mathscr{L} \in \mathscr{C}$
- family $\{i_X : X^* \otimes X \to \mathscr{L}\}_{X \in \mathscr{C}}$ of morphisms

Universal Property of Coends

Universal means for every dinatural transformation D of source H there exists a unique morphism $f_D : \mathscr{L} \to D$ such that the diagram



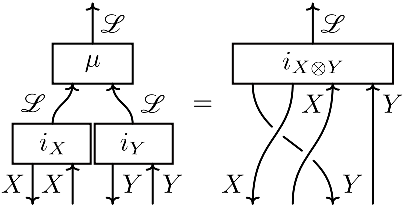
commutes for all $X, Y \in \mathscr{C}$, $f \in \mathscr{C}(X, Y)$

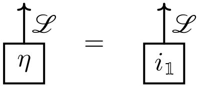
Existence and Properties of Coends

Proposition (Majid + Lyubashenko[11])

- *There exists a unique coend $\mathscr{L} \in \mathscr{C}$ up to isomorphism*
- *$\mathscr{L}$ is a braided Hopf algebra in $\mathscr{C}$, meaning it admits*
  - *product $\mu : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L}$, unit $\eta : \mathbb{1} \to \mathscr{L}$*
  - *coproduct $\Delta : \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$, counit $\varepsilon : \mathscr{L} \to \mathbb{1}$*
  - *antipode $S : \mathscr{L} \to \mathscr{L}$*

satisfying several axioms





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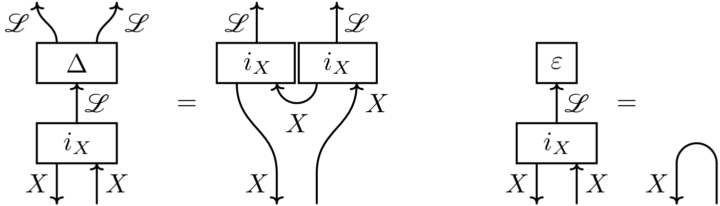

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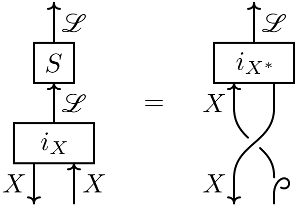


[11] See [M91, L95]

Example: Coend for $\bar{U}_q\mathfrak{sl}_2$

- $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$
- $\mathscr{L} = \mathrm{coad}$ *coadjoint representation*[12]
  - vector space $(\bar{U}_q\mathfrak{sl}_2)^*$
  - coadjoint $\bar{U}_q\mathfrak{sl}_2$-action given by

$$\mathrm{coad}_u(f) := f(S(u_{(1)})\_u_{(2)})$$

  for all $u \in \bar{U}_q\mathfrak{sl}_2$, $f \in (\bar{U}_q\mathfrak{sl}_2)^*$, where

$$\Delta(u) = u_{(1)} \otimes u_{(2)}$$

- $$\begin{aligned} i_X : X^* \otimes X &\to \mathscr{L} \\ (f,x) &\mapsto f(\_ \cdot x) \end{aligned}$$ for every $X \in \mathscr{C}$

[12] See [L94]

Integrals

Definition (Integral (of $\mathscr{L}$))

Morphism $\Lambda : \mathbb{1} \to \mathscr{L}$ satisfying

$$\mu \circ (\Lambda \otimes \mathrm{id}_{\mathscr{L}}) = \Lambda \circ \varepsilon = \mu \circ (\mathrm{id}_{\mathscr{L}} \otimes \Lambda)$$

Proposition (Lyubashenko[13])

There exists a unique integral $\Lambda : \mathbb{1} \to \mathscr{L}$ up to scalar

[13] See [L95]

Example: Integral of $\bar{U}_q\mathfrak{sl}_2$

- $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$
- $\mathscr{L} = \mathrm{coad}$
- $\{E^a F^b K^c \mid 0 \leqslant a, b, c \leqslant r-1\}$ basis of $\bar{U}_q\mathfrak{sl}_2$
- $\Lambda : \mathbb{C} \to \mathscr{L}$ determined by $\lambda := \Lambda(1) \in (\bar{U}_q\mathfrak{sl}_2)^*$ satisfying[14]

$$\lambda(E^a F^b K^c) := \delta_{a,r-1}\delta_{b,r-1}\delta_{c,1}$$

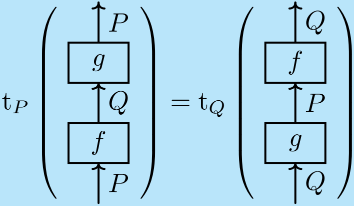
for all $0 \leqslant a, b, c \leqslant r-1$

[14]See [L94]

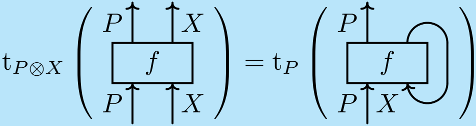
Modified Traces

Definition (Modified Trace (on $\mathrm{Proj}(\mathscr{C})$))

Family $\mathrm{t} := \{\mathrm{t}_P : \mathrm{End}_{\mathscr{C}}(P) \to \Bbbk\}_{P \in \mathrm{Proj}(\mathscr{C})}$ of linear maps satisfying

- 

  $\forall\, P, Q \in \mathrm{Proj}(\mathscr{C})$
  $\forall\, f \in \mathscr{C}(P, Q)$
  $\forall\, g \in \mathscr{C}(Q, P)$

- 

  $\forall\, P \in \mathrm{Proj}(\mathscr{C})$
  $\forall\, X \in \mathscr{C}$
  $\forall\, f \in \mathrm{End}_{\mathscr{C}}(P \otimes X)$

Existence and Non-Degeneracy of Modified Traces

- t modified trace on $\mathrm{Proj}(\mathscr{C})$
- t *non-degenerate* if

$$\begin{array}{rcl} \mathrm{t}_P(\_ \circ \_) : \mathscr{C}(X,P) \times \mathscr{C}(P,X) & \to & \Bbbk \\ (g,f) & \mapsto & \mathrm{t}_P(g \circ f) \end{array}$$

non-degenerate $\forall\, P \in \mathrm{Proj}(\mathscr{C}), X \in \mathscr{C}$

Proposition (Geer-Kujawa-Patureau[15])

There exists a unique modified trace t *on* $\mathrm{Proj}(\mathscr{C})$ *up to scalar, and furthermore* t *is non-degenerate*

[15] See [GKP18]

Example: Modified Trace for $\bar{U}_q\mathfrak{sl}_2$

- $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2$-mod
- $\bar{U} \in \mathrm{Proj}(\mathscr{C})$ regular representation
- $f_{a,b,c} \in \mathrm{End}_{\mathscr{C}}(\bar{U})$ determined by

$$f_{a,b,c}(1) := E^a F^b K^c$$

- t on $\mathrm{Proj}(\mathscr{C})$ determined by[16]

$$\mathrm{t}_{\bar{U}}(f_{a,b,c}) := \delta_{a,r-1}\delta_{b,r-1}\delta_{c,0}$$

for all $0 \leqslant a, b, c \leqslant r-1$

[16] See [BBG18]

Summary

Toolbox for TQFT construction

- Modular category $\mathscr{C}$
- Coend $\mathscr{L} \in \mathscr{C}$
- Integral $\Lambda : \mathbb{1} \to \mathscr{L}$
- Modified trace t on $\mathrm{Proj}(\mathscr{C})$

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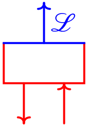
Bichrome Graphs

Bichrome graphs are ribbon graphs[17] with

- edges of two kinds
  - *red* (unlabeled)
  - *blue* (labeled by objects of $\mathscr{C}$)
- coupons of two kinds
  - *bichrome* (unlabeled)
  - *blue* (labeled by morphisms of $\mathscr{C}$)

satisfying

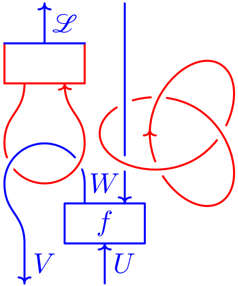
- bichrome coupons are all of the form
- boundary vertices are all blue



[17] See [T94]

Example of Bichrome Graph

If $U, V, W \in \mathscr{C}$ and $f \in \mathscr{C}(U, V \otimes W^*)$ then



is a bichrome graph from $((-, V), (+, U))$ to $((+, \mathscr{L}), (-, W))$

Category $\mathscr{R}_\Lambda$ of Bichrome Graphs

$\mathscr{R}_\Lambda$ category with

- obects: finite sequences $(\underline{\varepsilon}, \underline{V}) = ((\varepsilon_1, V_1), \dots, (\varepsilon_m, V_m))$
  - $\varepsilon_i \in \{+,-\} \quad \forall\, 1 \leqslant i \leqslant m$
  - $V_i \in \mathscr{C} \quad \forall\, 1 \leqslant i \leqslant m$
- morphisms: isotopy classes of bichrome graphs T with boundary vertices specified by source and target

$\mathscr{R}_\mathscr{C}$ category of ribbon graphs embedded

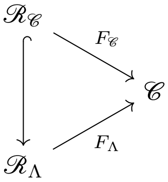
$$\mathscr{R}_\mathscr{C} \hookrightarrow \mathscr{R}_\Lambda$$

as subcategory of blue graphs

Lyubashenko's construction can be reformulated as a functor

$$F_\Lambda : \mathscr{R}_\Lambda \to \mathscr{C}$$

which fits into the commutative diagram

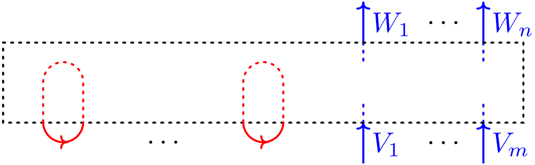


where $F_\mathscr{C}$ is the Reshetikhin-Turaev functor[18]

[18] See [T94]

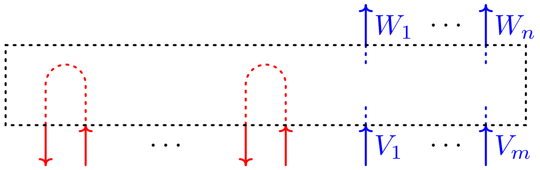
Algorithm for the Definition of F_Λ

- Consider T from $((+, V_1), \ldots, (+, V_m))$ to $((+, W_1), \ldots, (+, W_n))$



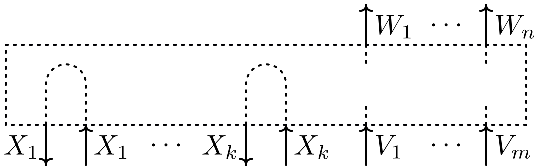
Algorithm for the Definition of F_Λ

- Consider T from $((+,V_1),\dots,(+,V_m))$ to $((+,W_1),\dots,(+,W_n))$
- Open red components to obtain a bottom graph $\tilde{T}$



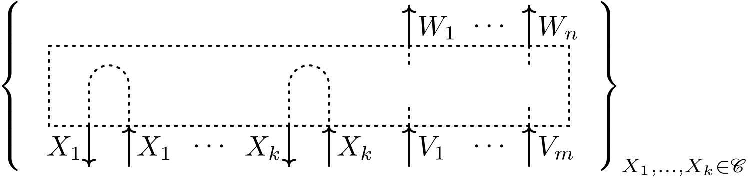
Algorithm for the Definition of F_Λ

- Consider T from $((+,V_1),\dots,(+,V_m))$ to $((+,W_1),\dots,(+,W_n))$
- Open red components to obtain a bottom graph $\tilde{T}$
- Label red components and forget difference with blue ones



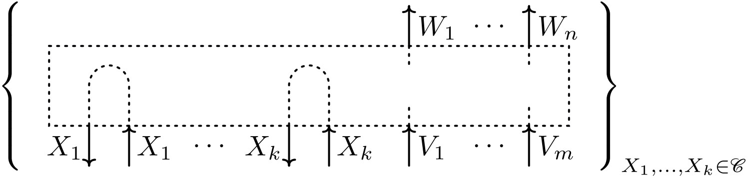
Algorithm for the Definition of F_Λ

- Consider T from $((+,V_1),\ldots,(+,V_m))$ to $((+,W_1),\ldots,(+,W_n))$
- Open red components to obtain a bottom graph $\tilde{T}$
- Label red components and forget difference with blue ones



Algorithm for the Definition of F_Λ

- Consider T from $((+,V_1),\ldots,(+,V_m))$ to $((+,W_1),\ldots,(+,W_n))$
- Open red components to obtain a bottom graph $\tilde{T}$
- Label red components and forget difference with blue ones
- Obtain $f_{\mathscr{C}}(\tilde{T}) : \mathscr{L}^{\otimes k} \otimes V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$



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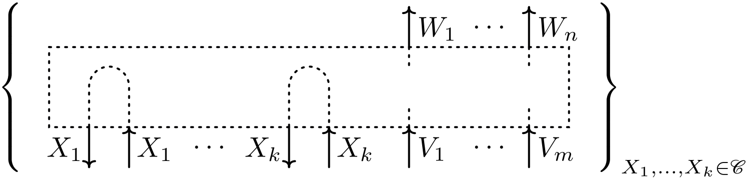
- Obtain $f_{\mathscr{C}}(\tilde{T}) : \mathscr{L}^{\otimes k} \otimes V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$
- Define[19] $F_\Lambda(T) : V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$ as

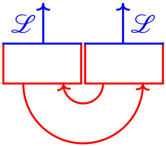
$$F_\Lambda(T) := f_{\mathscr{C}}(\tilde{T}) \circ (\Lambda^{\otimes k} \otimes \mathrm{id}_{V_1 \otimes \ldots \otimes V_m})$$



[19] Compare with [H05, DGP17]

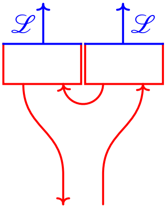
Example of Computation of F_Λ

- Consider T from $\varnothing$ to $((+,\mathscr{L}),(+,\mathscr{L}))$



Example of Computation of F_Λ

- Consider T from $\varnothing$ to $((+, \mathscr{L}), (+, \mathscr{L}))$
- Open red component to obtain a bottom graph $\tilde{T}$



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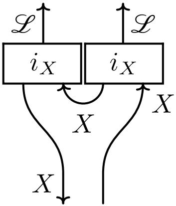

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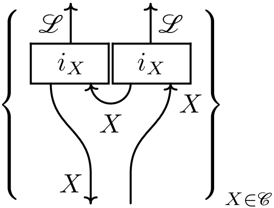
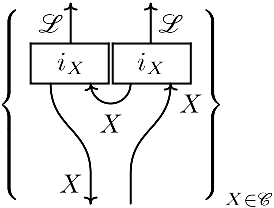


Example of Computation of F_Λ

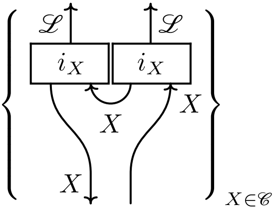
- Consider T from $\varnothing$ to $((+, \mathscr{L}), (+, \mathscr{L}))$
- Open red component to obtain a bottom graph $\tilde{T}$
- Label red component and forget difference with blue ones
- Obtain $\Delta : \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$



Example of Computation of F_Λ

- Consider T from $\varnothing$ to $((+,\mathscr{L}),(+,\mathscr{L}))$
- Open red component to obtain a bottom graph $\tilde{T}$
- Label red component and forget difference with blue ones
- Obtain $\Delta : \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$
- Compute $F_\Lambda(T) : \mathbb{1} \to \mathscr{L} \otimes \mathscr{L}$ as

$$F_\Lambda(T) = \Delta \circ \Lambda$$



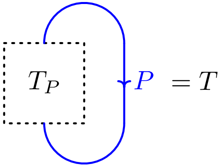
Admissible Closed Bichrome Graphs

T bichrome graph

- T *closed* if it has no boundary vertex
- T *admissible* if it has some $P \in \mathrm{Proj}(\mathscr{C})$ among its labels

T admissible closed bichrome graph, $P \in \mathrm{Proj}(\mathscr{C})$

- T_P from $(+, P)$ to itself *cutting presentation of* T if



Admissible Closed Bichrome Graph Invariant

Theorem

If T is an admissible closed bichrome graph, and T_P is a cutting presentation of T, then

$$F'_\Lambda(T) := \mathrm{t}_P(F_\Lambda(T_P))$$

is a topological invariant of T

Idea: the defining properties of the modified trace t ensure independence of the choice of a cutting presentation T_P

Surgery Presentations

Surgery

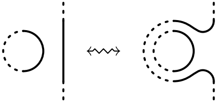
M 3-manifold, $L \subset M$ framed link $\rightsquigarrow M(L)$ 3-manifold

- Lickorish-Wallace theorem: For every closed connected 3-manifold M there exists a framed link $L \subset S^3$ such that

$$M \cong S^3(L)$$

- Kirby calculus: $S^3(L) \cong S^3(L') \Leftrightarrow L$ and L' are related by a sequence of Kirby moves

Kirby I



Kirby II

Stabilization Coefficients

- *Stabilization coefficients* are defined as

$$\Delta_+ := F_\Lambda\Big($$  $$\Big) \in \Bbbk \qquad \Delta_- := F_\Lambda\Big($$  $$\Big) \in \Bbbk$$

- $\mathscr{C}$ modular $\Rightarrow \Delta_+, \Delta_- \in \Bbbk^\times$
- Fix coefficients $\mathscr{D}, \delta \in \Bbbk^\times$ satisfying

$$\mathscr{D}^2 = \Delta_+ \Delta_- \qquad \delta = \frac{\Delta_+}{\mathscr{D}} = \frac{\mathscr{D}}{\Delta_-}$$

Example: $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$

$$\Delta_+ = i^{-\frac{r-1}{2}} r^{-\frac{3}{2}} q^{\frac{r-3}{2}} \{1\}^{2r-2} \qquad \Delta_- = i^{\frac{r-1}{2}} r^{-\frac{3}{2}} q^{\frac{r+3}{2}} \{1\}^{2r-2}$$

$$\mathscr{D} = r^{-\frac{3}{2}} \{1\}^{2r-2} \qquad \delta = i^{-\frac{r-1}{2}} q^{\frac{r-3}{2}}$$

Admissible Decorated 3-Manifold Invariant

Theorem

If M is a closed connected 3-manifold, $T \subset M$ is an admissible closed bichrome graph, and $L \subset S^3$ is a red surgery presentation of M with ℓ components and signature[20] σ, then

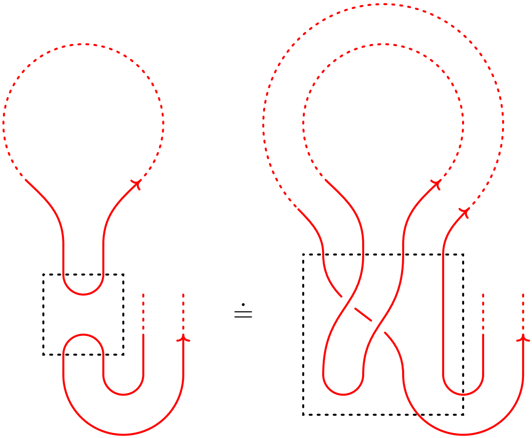
$$\mathrm{L}'_{\mathscr{C}}(M,T) := \mathscr{D}^{-1-\ell}\delta^{-\sigma}F'_{\Lambda}(L \cup T)$$

is a topological invariant of (M,T)

Idea: the defining properties of the integral Λ are an algebraic version of the invariance under Kirby II moves of the functor F_{Λ}

[20] See [T94]

Invariance Under Kirby II Moves



$\Lambda \circ \varepsilon \doteq \mu \circ (\Lambda \otimes \mathrm{id}_{\mathscr{L}})$

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Category $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$ of Admissible Decorated Cobordisms

$\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$ category with

- obects: triples $\mathbb{\Sigma} = (\Sigma, P, \lambda)$
  - Σ closed surface
  - $P \subset \Sigma$ oriented framed finite set labeled by objects of $\mathscr{C}$
  - $\lambda \subset H_1(\Sigma; \mathbb{R})$ Lagrangian subspace[21]
- morphisms: equivalence classes of admissible triples $\mathbb{M} = (M, T, n)$ from $\mathbb{\Sigma} = (\Sigma, P, \lambda)$ to $\mathbb{\Sigma}' = (\Sigma', P', \lambda')$
  - M 3-dimensional cobordism from Σ to Σ'
  - $T \subset M$ bichrome graph from P to P'
  - $n \in \mathbb{Z}$ integer (called *signature defect*)
- admissibility: every connected component of M disjoint from Σ contains an admissible subgraph of T

[21] See [T94]

Composition

- $\mathbb{M} : \mathbb{\Sigma} \to \mathbb{\Sigma}'$, $\mathbb{M}' : \mathbb{\Sigma}' \to \mathbb{\Sigma}''$ morphisms of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$
- $\mathbb{\Sigma} = (\Sigma, P, \lambda)$, $\mathbb{\Sigma}' = (\Sigma', P', \lambda')$, $\mathbb{\Sigma}'' = (\Sigma'', P'', \lambda'')$
- $\mathbb{M} = (M, T, n)$, $\mathbb{M}' = (M', T', n')$
- $\mathbb{M}' \circ \mathbb{M} : \mathbb{\Sigma} \to \mathbb{\Sigma}''$ defined as the equivalence class of

$$(M \cup_{\Sigma'} M', T \cup_{P'} T', n + n' - \mu\left(M_*(\lambda), \lambda', M'^*(\lambda'')\right))$$

  - $M_*(\lambda) \subset H_1(\Sigma'; \mathbb{R})$ push-forward of $\lambda \subset H_1(\Sigma; \mathbb{R})$ under M
  - $M'^*(\lambda'') \subset H_1(\Sigma'; \mathbb{R})$ pull-back of $\lambda'' \subset H_1(\Sigma''; \mathbb{R})$ under M'
  - μ Maslov index[22]

[22] See [T94]

Tensor Product

- $\mathbb{\Sigma} = (\Sigma, P, \lambda)$, $\mathbb{\Sigma}' = (\Sigma', P', \lambda')$ objects of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$
- $\mathbb{\Sigma} \otimes \mathbb{\Sigma}' = \mathbb{\Sigma} \sqcup \mathbb{\Sigma}'$ defined as

$$(\Sigma \sqcup \Sigma', P \sqcup P', \lambda \oplus \lambda')$$

- $\mathbb{M} = (M, T, n)$, $\mathbb{M}' = (M', T', n')$ morphisms of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$
- $\mathbb{M} \otimes \mathbb{M}' = \mathbb{M} \sqcup \mathbb{M}'$ defined as the equivalence class of

$$(M \sqcup M', T \sqcup T', n + n')$$

Extension of $\mathrm{L}'_{\mathscr{C}}$ to Closed Morphisms of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$

- $\mathbb{M} : \mathbb{\Sigma} \to \mathbb{\Sigma}'$ *closed* if $\mathbb{\Sigma} = \mathbb{\Sigma}' = \varnothing$
- $\mathbb{M} = (M, T, n)$ *connected* if M connected

If $\mathbb{M} = (M, T, n)$ closed connected then

$$\mathrm{L}'_{\mathscr{C}}(\mathbb{M}) := \delta^n \mathrm{L}'_{\mathscr{C}}(M, T)$$

If $\mathbb{M} = \mathbb{M}_1 \sqcup \ldots \sqcup \mathbb{M}_k$ with $\mathbb{M}_i$ closed connected $\forall\, 1 \leqslant i \leqslant k$ then

$$\mathrm{L}'_{\mathscr{C}}(\mathbb{M}) := \prod_{i=1}^{k} \mathrm{L}'_{\mathscr{C}}(\mathbb{M}_i)$$

Idea: extend $\mathrm{L}'_{\mathscr{C}}$ to a functor using the universal construction[23]

[23] See [BHMV95]

Universal Construction

Σ object of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$

- $\mathscr{V}(\Sigma)$ free vector space generated by morphisms $\mathbb{M}_\Sigma : \varnothing \to \Sigma$
- $\mathscr{V}'(\Sigma)$ free vector space generated by morphisms $\mathbb{M}'_\Sigma : \Sigma \to \varnothing$
- $$\begin{aligned} \langle \_,\_ \rangle_\Sigma : \mathscr{V}'(\Sigma) \times \mathscr{V}(\Sigma) &\to \Bbbk \\ (\mathbb{M}'_\Sigma, \mathbb{M}_\Sigma) &\mapsto \mathrm{L}'_{\mathscr{C}}(\mathbb{M}'_\Sigma \circ \mathbb{M}_\Sigma) \end{aligned}$$
- $\mathrm{V}_{\mathscr{C}}(\Sigma) := \mathscr{V}(\Sigma) / \operatorname{rad}_\mathrm{R} \langle \_,\_ \rangle_\Sigma$
- $\mathrm{V}'_{\mathscr{C}}(\Sigma) := \mathscr{V}'(\Sigma) / \operatorname{rad}_\mathrm{L} \langle \_,\_ \rangle_\Sigma$

$\mathbb{M} : \Sigma \to \Sigma'$ morphism of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$

- $$\begin{aligned} \mathrm{V}_{\mathscr{C}}(\mathbb{M}) : \mathrm{V}_{\mathscr{C}}(\Sigma) &\to \mathrm{V}_{\mathscr{C}}(\Sigma') \\ [\mathbb{M}_\Sigma] &\mapsto [\mathbb{M} \circ \mathbb{M}_\Sigma] \end{aligned}$$
- $$\begin{aligned} \mathrm{V}'_{\mathscr{C}}(\mathbb{M}) : \mathrm{V}'_{\mathscr{C}}(\Sigma') &\to \mathrm{V}'_{\mathscr{C}}(\Sigma) \\ [\mathbb{M}'_{\Sigma'}] &\mapsto [\mathbb{M}'_{\Sigma'} \circ \mathbb{M}] \end{aligned}$$

$\mathrm{V}_{\mathscr{C}}$ and $\mathrm{V}'_{\mathscr{C}}$ are dual to each other in the sense that

$$\langle \mathrm{V}'_{\mathscr{C}}(\mathbb{M})[\mathbb{M}'_{\Sigma'}], [\mathbb{M}_{\Sigma}] \rangle_{\Sigma} = \langle [\mathbb{M}'_{\Sigma'}], \mathrm{V}_{\mathscr{C}}(\mathbb{M})[\mathbb{M}_{\Sigma}] \rangle_{\Sigma'}$$

for all vectors $[\mathbb{M}_{\Sigma}] \in \mathrm{V}_{\mathscr{C}}(\mathbb{\Sigma})$ and $[\mathbb{M}'_{\Sigma'}] \in \mathrm{V}'_{\mathscr{C}}(\mathbb{\Sigma}')$ and all morphisms $\mathbb{M} : \mathbb{\Sigma} \to \mathbb{\Sigma}'$

Theorem

The functors

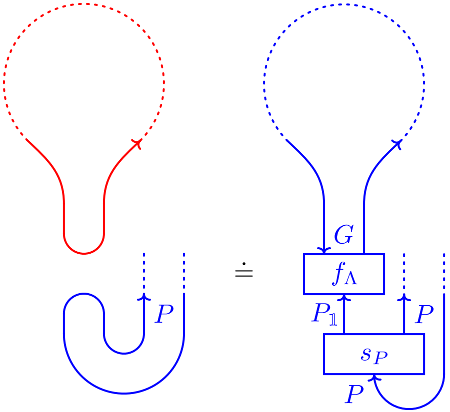
$$\mathrm{V}_{\mathscr{C}} : \check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}} \to \mathrm{Vect}_{\Bbbk} \qquad \mathrm{V}'_{\mathscr{C}} : (\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}})^{\mathrm{op}} \to \mathrm{Vect}_{\Bbbk}$$

are symmetric monoidal

Red Turns Blue

Red Turns Blue

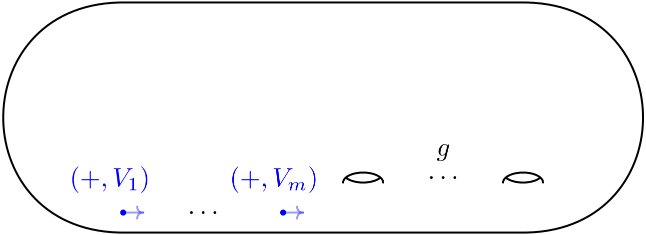
- $P_{\mathbb{1}}$ projective cover of $\mathbb{1}$
- $G \in \mathrm{Proj}(\mathscr{C})$ *projective generator*, $P \in \mathrm{Proj}(\mathscr{C})$ arbitrary
- There exist $f_\Lambda : P_{\mathbb{1}} \to G^* \otimes G$ and $s_P : P \to P_{\mathbb{1}} \otimes P$ such that



Algebraic Model for State Spaces

- $g \in \mathbb{N}$ and $\underline{V} = (V_1, \ldots, V_m) \in \mathscr{C}^{\times m}$
- $\mathbb{\Sigma}_{g,\underline{V}} = (\Sigma_g, P_{\underline{V}}, \lambda_g)$ object of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$

$(\Sigma_g, P_{\underline{V}}) =$



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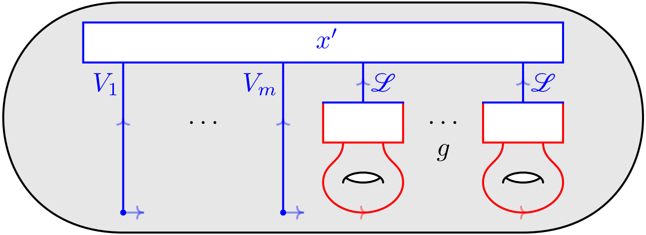
- $\Sigma_{g,\underline{V}} = (\Sigma_g, P_{\underline{V}}, \lambda_g)$ object of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$
- $\mathrm{X}'_{g,\underline{V}} := \mathscr{C}(V_1 \otimes \ldots \otimes V_m \otimes \mathscr{L}^{\otimes g}, \mathbb{1})$
- $$\begin{aligned} \Phi' : \mathrm{X}'_{g,\underline{V}} &\to \mathrm{V}'_{\mathscr{C}}(\Sigma_{g,\underline{V}}) \\ x' &\mapsto [M'_g, T_{x'}, 0] \end{aligned}$$

$(M'_g, T_{x'}) =$



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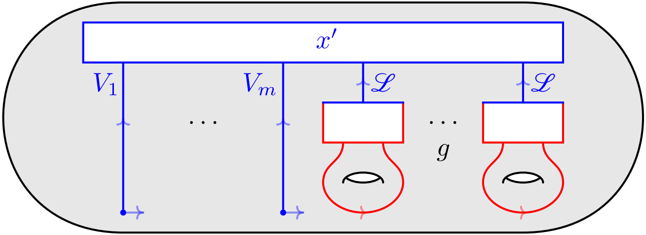
- $\begin{array}{rcl} \Phi' : \mathrm{X}'_{g,\underline{V}} & \to & \mathrm{V}'_{\mathscr{C}}(\Sigma_{g,\underline{V}}) \\ x' & \mapsto & [M'_g, T_{x'}, 0] \end{array}$ is an isomorphism

$(M'_g, T_{x'}) =$ 

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Mapping Cylinders

- $\mathbb{\Sigma} = (\Sigma, P, \lambda)$ object of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$
- $\mathrm{Diff}(\Sigma, P) := \{f \in \mathrm{Diff}(\Sigma) \mid f(P) = P\}$
- $\mathrm{Diff}_0(\Sigma, P) \triangleleft \mathrm{Diff}(\Sigma, P)$ subgroup of elements isotopic to id
- $\mathrm{Mod}(\Sigma, P) := \mathrm{Diff}(\Sigma, P) / \mathrm{Diff}_0(\Sigma, P)$
- $\mathbb{\Sigma} \times \mathbb{I}_f : \mathbb{\Sigma} \to \mathbb{\Sigma}$ *mapping cylinder of* $f \in \mathrm{Diff}(\Sigma, P)$ defined as

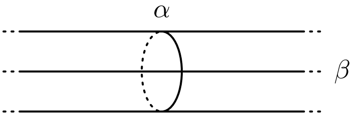
$$(\Sigma \times I_f, P \times I, 0)$$

for the cobordism $\Sigma \times I_f$ given by

  - 3-manifold $\Sigma \times I$
  - incoming boundary identification $f : \Sigma \to \Sigma \times \{0\}$
  - outgoing boundary identification $\mathrm{id} : \Sigma \to \Sigma \times \{1\}$

Dehn Twists and Curve Cylinders

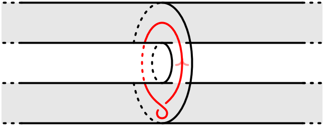
- $\tau_\alpha \in \mathrm{Diff}(\Sigma, P)$ *Dehn twist along simple closed curve* $\alpha \subset \Sigma$



- $\mathbb{\Sigma} \times \mathbb{I}_{\alpha_+} : \mathbb{\Sigma} \to \mathbb{\Sigma}$ *positive curve cylinder of* α defined as

$$(\Sigma \times I, (P \times I) \cup \alpha_+, 0)$$

for the framed red knot α_+ given by



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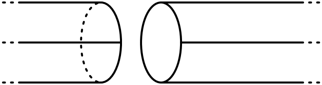

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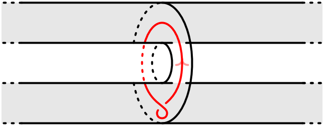

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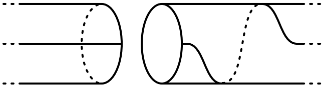

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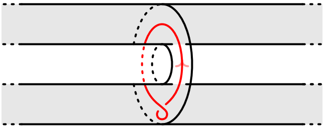

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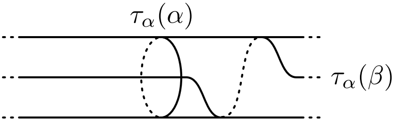

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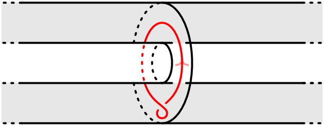
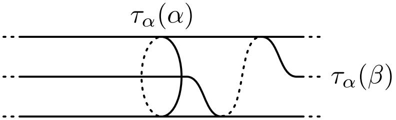


Dehn Twists and Curve Cylinders

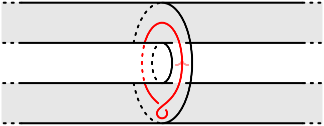
- $\tau_\alpha \in \mathrm{Diff}(\Sigma, P)$ *Dehn twist along simple closed curve* $\alpha \subset \Sigma$



- $\mathbb{\Sigma} \times \mathbb{I}_{\alpha_-} : \mathbb{\Sigma} \to \mathbb{\Sigma}$ *negative curve cylinder of* α defined as

$$(\Sigma \times I, (P \times I) \cup \alpha_-, 0)$$

for the framed red knot α_- given by



Quantum Representations

- $$\begin{aligned} \rho_{\mathscr{C}} : \mathrm{Mod}(\Sigma, P) &\to \mathrm{GL}_{\Bbbk}(\mathrm{V}_{\mathscr{C}}(\Sigma)) \\ [f] &\mapsto \mathrm{V}_{\mathscr{C}}(\Sigma \times \mathbb{I}_f) \end{aligned}$$ not homomorphism
- $$\begin{aligned} \rho'_{\mathscr{C}} : \mathrm{Mod}(\Sigma, P) &\to \mathrm{GL}_{\Bbbk}(\mathrm{V}'_{\mathscr{C}}(\Sigma)) \\ [f] &\mapsto \mathrm{V}'_{\mathscr{C}}(\Sigma \times \mathbb{I}_{f^{-1}}) \end{aligned}$$ not homomorphism
- $\rho_{\mathscr{C}}(\tau_\alpha) = \mathrm{V}_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\tau_\alpha}) \propto \mathrm{V}_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\alpha_-})$
- $\rho'_{\mathscr{C}}(\tau_\alpha) = \mathrm{V}'_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\tau_\alpha^{-1}}) \propto \mathrm{V}'_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\alpha_+})$

Quantum Representations

- $\bar{\rho}_{\mathscr{C}} : \mathrm{Mod}(\Sigma, P) \to \mathrm{PGL}_{\Bbbk}(\mathrm{V}_{\mathscr{C}}(\Sigma))$ homomorphism
  $[f] \mapsto [\mathrm{V}_{\mathscr{C}}(\Sigma \times \mathbb{I}_f)]$
- $\bar{\rho}'_{\mathscr{C}} : \mathrm{Mod}(\Sigma, P) \to \mathrm{PGL}_{\Bbbk}(\mathrm{V}'_{\mathscr{C}}(\Sigma))$ homomorphism
  $[f] \mapsto [\mathrm{V}'_{\mathscr{C}}(\Sigma \times \mathbb{I}_{f^{-1}})]$
- $\bar{\rho}_{\mathscr{C}}(\tau_\alpha) = [\mathrm{V}_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\tau_\alpha})] = [\mathrm{V}_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\alpha_-})]$
- $\bar{\rho}'_{\mathscr{C}}(\tau_\alpha) = [\mathrm{V}'_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\tau_\alpha^{-1}})] = [\mathrm{V}'_{\mathscr{C}}(\Sigma \times \mathbb{I}_{\alpha_+})]$

Equivalence With Lyubashenko

- $X'_{g,\underline{V}} := \mathscr{C}(V_1 \otimes \ldots \otimes V_m \otimes \mathscr{L}^{\otimes g}, \mathbb{1})$
- $\Phi' : X'_{g,\underline{V}} \to V'_{\mathscr{C}}(\Sigma_{g,\underline{V}})$ isomorphism
- $\bar{\rho}'_{\mathscr{C}} : \mathrm{Mod}(\Sigma_g, P_{\underline{V}}) \to \mathrm{PGL}_{\Bbbk}(X'_{g,\underline{V}})$ induced by Φ'
- $L_{g,\underline{V}} := \mathscr{C}(V_1 \otimes \ldots \otimes V_m, \mathscr{L}^{\otimes g})$
- $\bar{\rho}_{L} : \mathrm{Mod}(\Sigma_g, P_{\underline{V}}) \to \mathrm{PGL}_{\Bbbk}(L_{g,\underline{V}})$ Lyubashenko's representation

Theorem

$\bar{\rho}'_{\mathscr{C}}$ is equivalent to $\bar{\rho}_{L}$

Idea: compare the action of a set of generators using

$$\varphi : X'_{g,\underline{V}} \to L_{g,\underline{V}}$$

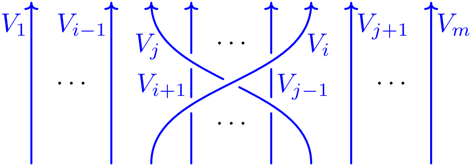
induced by $\mathscr{R} : \mathbb{1} \to \mathscr{L} \otimes \mathscr{L}$ for

$$\mathscr{R} := \Delta \circ \Lambda$$

Generators of Mapping Class Groups

- $\mathrm{B}(\Sigma_g, P_{\underline{V}})$ decorated surface framed braid group
- $1 \to \mathrm{B}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g) \to 1$ exact
- $\mathrm{Mod}(\Sigma_g, P_{\underline{V}})$ generated by
  - framed braids $x_{i,j}, w_{i,j}, v_i$
  - Dehn twists $\tau_{\alpha_k}, \tau_{\beta_j}, \tau_{\gamma_j}, \tau_{\delta_{i,j}}$

$x_{i,j} :=$

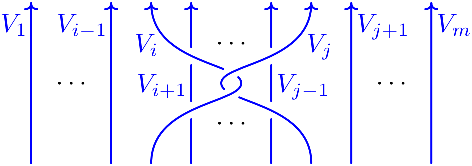


$V_i = V_j$

Generators of Mapping Class Groups

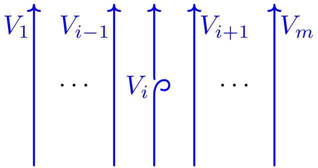
- $\mathrm{B}(\Sigma_g, P_{\underline{V}})$ decorated surface framed braid group
- $1 \to \mathrm{B}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g) \to 1$ exact
- $\mathrm{Mod}(\Sigma_g, P_{\underline{V}})$ generated by
  - framed braids $x_{i,j}, w_{i,j}, v_i$
  - Dehn twists $\tau_{\alpha_k}, \tau_{\beta_j}, \tau_{\gamma_j}, \tau_{\delta_{i,j}}$

$w_{i,j} :=$



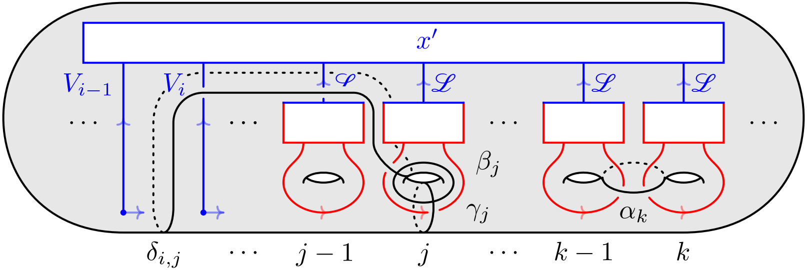
Generators of Mapping Class Groups

- $\mathrm{B}(\Sigma_g, P_{\underline{V}})$ decorated surface framed braid group
- $1 \to \mathrm{B}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g) \to 1$ exact
- $\mathrm{Mod}(\Sigma_g, P_{\underline{V}})$ generated by
  - framed braids $x_{i,j}, w_{i,j}, v_i$
  - Dehn twists $\tau_{\alpha_k}, \tau_{\beta_j}, \tau_{\gamma_j}, \tau_{\delta_{i,j}}$

$v_i :=$ 

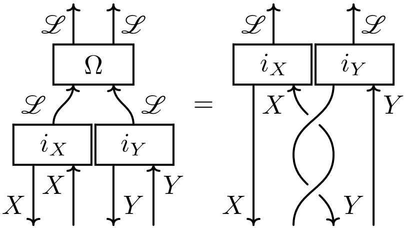
Generators of Mapping Class Groups

- $\mathrm{B}(\Sigma_g, P_{\underline{V}})$ decorated surface framed braid group
- $1 \to \mathrm{B}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g, P_{\underline{V}}) \to \mathrm{Mod}(\Sigma_g) \to 1$ exact
- $\mathrm{Mod}(\Sigma_g, P_{\underline{V}})$ generated by
  - framed braids $x_{i,j}, w_{i,j}, v_i$
  - Dehn twists $\tau_{\alpha_k}, \tau_{\beta_j}, \tau_{\gamma_j}, \tau_{\delta_{i,j}}$



Example: Action of $S_j := \tau_{\gamma_j} \circ \tau_{\beta_j} \circ \tau_{\gamma_j}$

- $\Omega : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$ *monodromy*

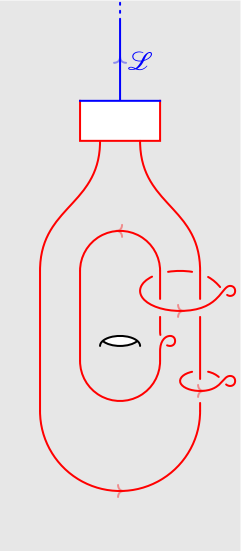


- $\mathscr{S} : \mathscr{L} \to \mathscr{L}$ *S-transformation*

$$\mathscr{S} := (\varepsilon \otimes \mathrm{id}_{\mathscr{L}}) \circ \Omega \circ (\mathrm{id}_{\mathscr{L}} \otimes \Lambda)$$

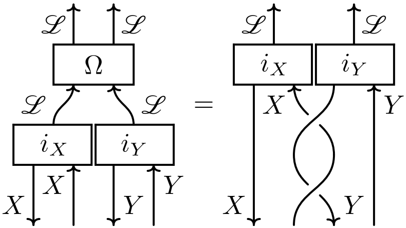
- $\rho'_{\mathscr{C}}(S_j) \propto (\_ \circ \mathscr{S}_j)$ where

$$\mathscr{S}_j = \mathrm{id}_{V_1 \otimes \ldots \otimes V_m \otimes \mathscr{L}^{\otimes j-1}} \otimes \mathscr{S} \otimes \mathrm{id}_{\mathscr{L}^{\otimes g-j}}$$



Example: Action of $S_j := \tau_{\gamma_j} \circ \tau_{\beta_j} \circ \tau_{\gamma_j}$

- $\Omega : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$ *monodromy*



- $\mathscr{S} : \mathscr{L} \to \mathscr{L}$ *S-transformation*

$$\mathscr{S} := (\varepsilon \otimes \mathrm{id}_{\mathscr{L}}) \circ \Omega \circ (\mathrm{id}_{\mathscr{L}} \otimes \Lambda)$$

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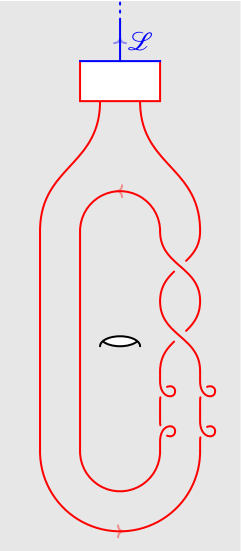
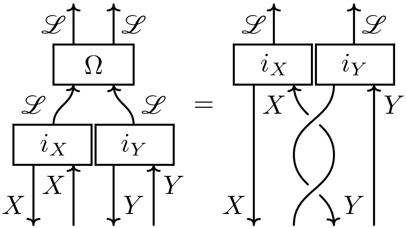


Example: Action of $S_j := \tau_{\gamma_j} \circ \tau_{\beta_j} \circ \tau_{\gamma_j}$

- $\Omega : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$ *monodromy*



- $\mathscr{S} : \mathscr{L} \to \mathscr{L}$ *S-transformation*

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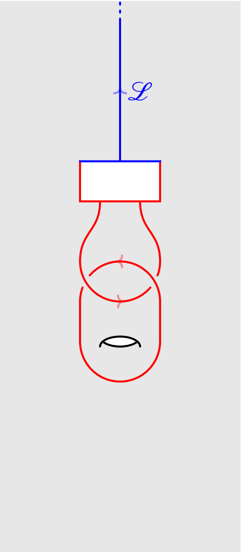
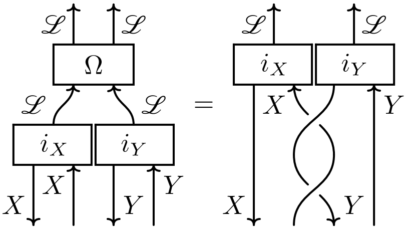


Example: Action of $S_j := \tau_{\gamma_j} \circ \tau_{\beta_j} \circ \tau_{\gamma_j}$

- $\Omega : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}$ *monodromy*

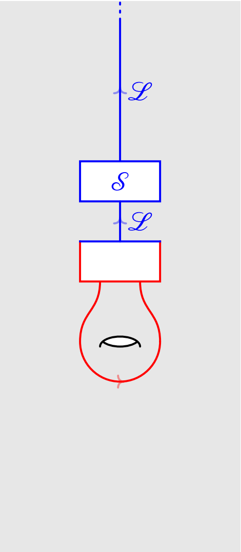


- $\mathscr{S} : \mathscr{L} \to \mathscr{L}$ *S-transformation*

$$\mathscr{S} := (\varepsilon \otimes \mathrm{id}_{\mathscr{L}}) \circ \Omega \circ (\mathrm{id}_{\mathscr{L}} \otimes \Lambda)$$

- $\rho'_{\mathscr{C}}(S_j) \propto (\_ \circ \mathscr{S}_j)$ where

$$\mathscr{S}_j = \mathrm{id}_{V_1 \otimes \ldots \otimes V_m \otimes \mathscr{L}^{\otimes j-1}} \otimes \mathscr{S} \otimes \mathrm{id}_{\mathscr{L}^{\otimes g-j}}$$



Infinite-Order Dehn Twists

- $\mathscr{C} = \bar{U}_q\mathfrak{sl}_2\text{-mod}$
- $\mathbb{\Sigma} = (\Sigma, \varnothing, \lambda)$ object of $\check{\mathrm{C}}\mathrm{ob}_{\mathscr{C}}$
- $\alpha \subset \Sigma$ simple closed curve

Proposition

$\bar{\rho}_{\mathscr{C}}(\tau_\alpha)$ *has infinite order in* $\mathrm{PGL}_{\Bbbk}(\mathrm{V}_{\mathscr{C}}(\mathbb{\Sigma}))$

Semisimple vs Non-Semisimple

This never happens for $\bar{\mathscr{C}} = \mathscr{C}/\mathscr{N}(\mathscr{C})$

References I

[A88] M. Atiyah, *Topological Quantum Field Theories*, Publ. Math. Inst. Hautes Études Sci. **68** (1988), 175-186.

[BBG17] A. Beliakova, C. Blanchet, N. Geer, *Logarithmic Hennings Invariants for Restricted Quantum* $\mathfrak{sl}(2)$, Algebr. Geom. Topol. **18** (2018), No. 7, 4329-4358; arXiv:1705.03083 [math.GT].

[BBG18] A. Beliakova, C. Blanchet, A. Gainutdinov, *Modified Trace is a Symmetrised Integral*, arXiv:1801.00321 [math.QA].

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel, *Topological Quantum Field Theories Derived from the Kauffman Bracket*, Topology **34** (1995), No. 4, 883-927.

[CGP12] F. Costantino, N. Geer, B. Patureau-Mirand, *Quantum Invariants of 3-Manifolds via Link Surgery Presentations and Non-Semi-Simple Categories*, J. Topol. **7** (2014), No. 4, 1005-1053; arXiv:1202.3553 [math.GT].

[CKS07] Q. Chen, S. Kuppum, P. Srinivasan, *On the Relation Between the WRT Invariant and the Hennings Invariant*, Math. Proc. Cambridge Philos. Soc. **146** (2009), No. 1, 151-163; arXiv:0709.2318 [math.QA].

References II

[DGGPR19] M. De Renzi, A. Gainutdinov, N. Geer, B. Patureau-Mirand, I. Runkel, *3-Dimensional TQFTs from Non-Semisimple Modular Categories*, arXiv:1912.02063 [math.GT].

[DGGPR20] M. De Renzi, A. Gainutdinov, N. Geer, B. Patureau-Mirand, I. Runkel, *Mapping Class Group Representations from Non-Semisimple TQFTs*, In Preparation.

[DGP17] M. De Renzi, N. Geer, B. Patureau, *Renormalized Hennings Invariants and 2+1-TQFTs*, Comm. Math. Phys. **362** (2018), No. 3, 855-907; arXiv:1707.08044 [math.GT].

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor Categories*, Math. Surveys Monogr. **205**, Amer. Math. Soc., Providence, RI, 2015.

[GKP18] N. Geer, J. Kujawa, B. Patureau-Mirand, *M-Traces in (Non-Unimodular) Pivotal Categories*, arXiv:1809.00499 [math.RT].

[H96] M. Hennings, *Invariants of Links and 3-Manifolds Obtained from Hopf Algebras*, J. London Math. Soc. (2) **54** (1996), No. 3, 594-624.

[H05] K. Habiro, *Bottom Tangles and Universal Invariants*, Algebr. Geom. Topol. **6** (2006), No. 3, 1113--1214; arXiv:math/0505219.

References III

[K96] T. Kerler, *On the Connectivity of Cobordisms and Half-Projective TQFT's*, Comm. Math. Phys. **198** (1998), No. 3, 535-590; arXiv:q-alg/9603017.

[KL01] T. Kerler, V. Lyubashenko, *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, Lecture Notes in Mathematics **1765**. Springer-Verlag, Berlin, 2001.

[L90] G. Lusztig, *Finite Dimensional Hopf Algebras Arising from Quantized Universal Enveloping Algebras*, J. Amer. Math. Soc. **3** (1990), No. 1, 257--296.

[L93] G. Lusztig, *Introduction to Quantum Groups*, Progress in Mathematics **110**, Birkhäuser Boston, Inc., Boston, MA, 1993.

[L94] V. Lyubashenko, *Invariants of 3-Manifolds and Projective Representations of Mapping Class Groups via Quantum Groups at Roots of Unity*, Comm. Math. Phys. **172** (1995), No. 3, 467-516; arXiv:hep-th/9405167.

[L95] V. Lyubashenko, *Modular Transformations for Tensor Categories*, J. Pure Appl. Algebra **98** (1995), No. 3, 279-327.

[M13] J. Murakami, *Generalized Kashaev Invariants for Knots in Three Manifolds*, Quantum Topol. **8** (2017), No. 1, 35-73; arXiv:1312.0330 [math.GT].

References IV

[M91] S. Majid, *Reconstruction Theorems and Rational Conformal Field Theories*, Internat. J. Modern Phys. A **6** (1991), No. 24, 4359-4374.

[M95] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.

[O95] T. Ohtsuki, *Invariants of 3-Manifolds Derived from Universal Invariants of Framed Links*, Math. Proc. Cambridge Philos. Soc. **117** (1995), No. 2, 259-273.

[RT91] N. Reshetikhin, V. Turaev, *Invariants of 3-Manifolds via Link Polynomials and Quantum Groups*, Invent. Math. **103** (1991), No. 1, 547-597.

[S16] K. Shimizu, *Non-Degeneracy Conditions for Braided Finite Tensor Categories*, Adv. Math. **355** (2019), 106778; `arXiv:1602.06534` [math.QA].

[T94] V. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, De Gruyter Studies in Mathematics **18**, Walter de Gruyter & Co., Berlin, 1994.

[W89] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Comm. Math. Phys. **121** (1989), No. 3, 351-399.