Product Set Growth and Hyperbolic Geometry

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Product sets

G group, X δ–hyperbolic metric space, $G \curvearrowright X$

$U \subset G$ finite, $|U| =$ cardinality of U

$$U^n = \{g \mid g = u_1 \cdots u_n,\ u_i \in U\},\ U^n \subset G$$

Product sets

$G = \mathbb{Z} = \langle t \rangle,\ U = \{t, 1, t^{-1}\}$

$|U^n| = 2n + 1 \leqslant n|U|$

$G = \mathbb{F}_2 = \langle a, b \rangle,\ U = \{a, b, 1, a^{-1}, b^{-1}\}$

$|U^n| = 2 \cdot 3^n - 1$ and $|U^n| > |U|^{n/2}$

Product sets in free groups

Theorem(Razborov,Safin)

$G = \mathbb{F}_2$, $\exists \alpha > 0$ such that for all finite $U \subset G$, if $\langle U \rangle$ not cyclic,

$$|U^3| > (\alpha |U|)^2$$

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Product sets in hyperbolic groups

Theorem

G hyperbolic, $\exists \alpha > 0$ such that for all finite $U \subset G$, if $\langle U \rangle$ not virtually cyclic,

$$|U^3| > (\alpha |U|)^2 \text{ and in general } |U^n| > (\alpha |U|)^{\lfloor (n+1)/2 \rfloor}.$$

$$\Longrightarrow h(U) = \lim_{n\to\infty} \frac{1}{n} \log |U^n| \geqslant \frac{1}{2} \log(\alpha |U|)$$

Product sets in hyperbolic groups

$\delta > 0$, X δ–hyperbolic graph, $G \curvearrowright X$ properly cocompactly

- $\inf_{x \in X} d(x, gx) > 10^8\delta$ for all $g \neq 1 \in G$:

$$\alpha = \frac{1}{1200} \cdot \frac{1}{|B(x_0, 1000\delta)|^2}$$

- in general, $b := |\{g \mid d(x, gx) \leqslant 10^8\delta\}|$, and

$$\alpha = \frac{1}{10^{18}} \cdot \frac{1}{b^5} \cdot \frac{1}{|B(x_0, 1000\delta)|^2}$$

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Product sets in hyperbolic groups

$G = \mathbb{F}_2 = \langle a, b \rangle$

$U_N := \{a^{-N}, \ldots, a^{-1}, 1, a, \ldots, a^N\} \cup \{b\}$

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Proposition (Button)

Let G be a group. If $\exists c > 0$, $\varepsilon > 0$ such that

for all $U \subset G$ with $\langle U \rangle$ not virtually nilpotent

$$|U^3| > c|U|^{2+\varepsilon},$$

then either G has bounded exponent or is locally virtually nilpotent.

Product sets in hyperbolic groups

Theorem

G hyperbolic, $\exists\alpha > 0$ such that for all finite $U \subset G$, if $\langle U \rangle$ not virtually cyclic,

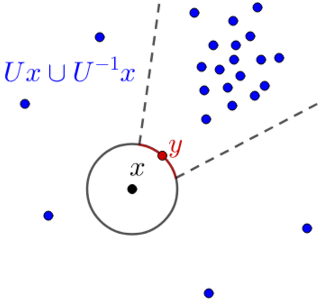
$$|U^3| > (\alpha|U|)^2 \text{ and in general } |U^n| > (\alpha|U|)^{\lfloor (n+1)/2 \rfloor}.$$

Actions on trees

free group, $A * B$, $A *_C B$

Energy and maximal displacement of $U \subset G$

1. $E(U) := 1/|U| \sum_{u \in U} d(x, ux)$
2. $x_0 =$ minimiser for $E(U)$
3. $\lambda_0(U) := \max_{u \in U} d(x_0, ux_0)$



Remarks

- If $E(U) > \varepsilon + \delta$, then $\lambda_0(U) \geqslant \varepsilon$.
- $G = A * \mathbb{Z}, U = U_0 \cup \{t\}$. For large n:

  $E(U) = 1/n$, but $\lambda_0(U) > n$.

Groups acting on trees

Theorem

$\forall \rho_0$, $k \geqslant \rho_0/10^{11}$, $\exists \alpha > 0$ such that

for all G acting k-acylindrical on tree of edge-length ρ_0,

for all finite $U \subset G$ with $\lambda_0(U) \geqslant 10^{11}k$ and if $\langle U \rangle$ not infinite cyclic,

$$|U^3| > (\alpha|U|)^2 \text{ and in general } |U^n| > (\alpha|U|)^{\lfloor (n+1)/2 \rfloor}.$$

Groups acting on trees

$\rho_0,\ k \geqslant \rho_0/10^{11}$

G acting k-acylindrical on tree of edge-length ρ_0

- $\alpha = \frac{\rho_0}{10^{24}k}$
- $G = A * B$, $k = \rho_0/10^{11}$. If U not conjugate into vertex stabiliser, $\lambda_0(U) \geqslant 10^{11}k$.

Groups acting on hyperbolic spaces

Theorem

$\forall \delta > 0,\ k_0 \geqslant \delta,\ n_0 > 0,\ \exists \varepsilon, \alpha > 0$ such that

for all G acting (k_0, n_0)-acylindrical on X,

for all finite $U \subset G$ with $\lambda_0(U) > \varepsilon \log(2|U|)$, if $\langle U \rangle$ not virtually cyclic,

$$|U^3| > (\frac{\alpha}{\log(2|U|)}|U|)^2 \text{ and in general } |U^n| > (\frac{\alpha}{\log(2|U|)}|U|)^{\lfloor (n+1)/2 \rfloor}.$$

Groups acting on hyperbolic spaces

$\delta > 0,\ k_0 \geqslant \delta,\ n_0 > 0$

G acts (k_0, n_0)-acylindrical on X

- $\varepsilon = 10^{11} k_0,\ \alpha = \frac{1}{10^{20} n_0^5} \cdot \frac{\delta}{k_0^2}$

Proof

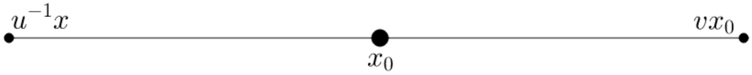
1. Reduction
   - Reduced products
   - Reduction Lemma
2. Periodicity
   - Periodic isometries
   - Construction of periodic isometries
3. Small cancellation and counting

Step 1. Reduced products

X tree, u, v isometries of X

Definition

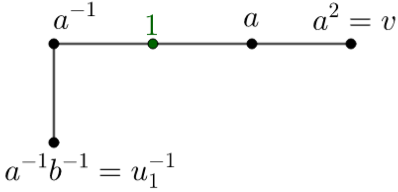
uv is reduced at $x_0 \in X$ if

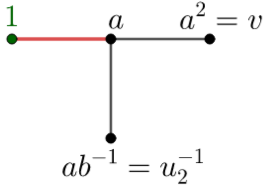


Step 1. Reduced products

$G = \mathbb{F}_2 = \langle a, b \rangle$, X Cayley graph, $x_0 = 1$

$u_1 = ba$, $u_2 = ba^{-1}$, $v = a^2$





Step 1. Reduced products and energy

$U \subset G$ finite, $x_0 :=$ minimiser for $E(U)$, $b := |B(x_0, 1000\delta)|$.

Reduction Lemma

$\exists\ U_0,\ U_1 \subseteq U$ such that

1. $|U_0| \geqslant 1/100b^2\ |U|$ and $|U_1| \geqslant 1/100b^2\ |U|$,
2. $U_0 U_1$ and $U_1 U_0$ reduced at x_0,
3. $1000\delta < |u_0 x_0 - x_0| \leqslant |u_1 x_0 - x_0|$.

Step 1. Reduced products and energy

$U \subset G$ finite, U_0, $U_1 \subset U$ given by Reduction Lemma

Reduction:

if $v \in U_1$, $|U^3| \geqslant |U_0 U_1 U_0| \geqslant |U_0 v U_0|$.

Count number of equations

$$u_1 v w_1 = u_2 v w_2 = \ldots = u_n v w_n, \quad u_i, w_i \in U_0.$$

Step 2. Periods

$G = \mathbb{F}_2$, $X =$ Cayley graph, $[g] := \inf_{x \in X} |gx - x|$,

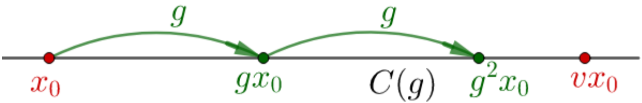
$C(g) := \{x \in X \mid |gx - x| < [g] + 20\delta\}$,

$E(g)$ maximal cyclic subgroup containing g

Definition

v is $E(g)$–periodic at x_0 if $x_0, vx_0 \in C(g)$ and

$$|x_0 - vx_0| > 2[g].$$



Step 2. Periods

$G = \mathbb{F}_2$, $X =$ Cayley graph, $x_0 = 1$

- periodic at x_0:

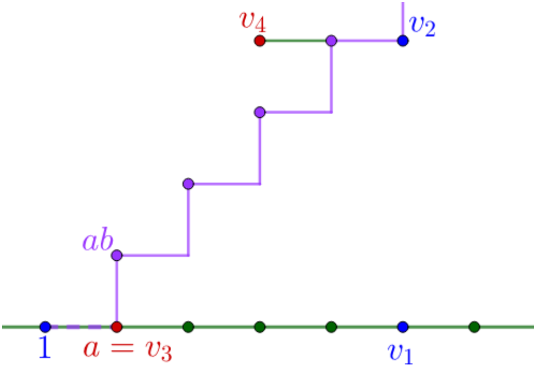
  $v_1 = aaaaa$

  $v_2 = abababababa$

- not periodic at x_0:

  $v_3 = a$

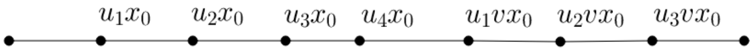
  $v_4 = abababab a^{-1}$



Step 2. Periods

$U \subset G = \mathbb{F}_2$, $X =$ Cayley graph

$u_1 v w_1 = u_2 v w_2 = \ldots = u_n v w_n, \quad u_i, w_i \in U_0, v \in U_1$



Lemma (Finding periods)

$n \geqslant 3$, $g := u_i^{-1} u_{i+1}$ such that $[u_i^{-1} u_{i+1}]$ is minimal.

Then:

$$v \text{ is } E(g)\text{–periodic at } x_0.$$

Step 3. Small cancellation and counting

$U \subset G = \mathbb{F}_2$, U_0, U_1 given by Reduction Lemma

$n \geqslant 3$

Corollary

$v \in U_1$. Then:

$$|U_0 v U_0| > 1/n |U_0|^2$$

unless v is $E(u_0^{-1} u_0')$–periodic at x_0.

Step 3. Small cancellation and counting

$U \subset G = \mathbb{F}_2$, U_0, U_1 given by Reduction Lemma

$n \geqslant 3$

Lemma

If $|U_0 U_1 U_0| \leqslant 1/4n|U_0|^2$, then for all v, $v' \in U_1$:

$$E(g) = E(g').$$

$\Longrightarrow U_1 \subset E$. If $\langle U \rangle$ not cyclic, use ping-pong argument:

$$\exists t \in U,\ t \notin E:\ |U^3| \geqslant |U_1 t U_1 t^{-1}| \geqslant |U_1|^2.$$

Thank you for your attention!

Theorem

G hyperbolic, $\exists \alpha > 0$ such that for all finite $U \subset G$, if $\langle U \rangle$ not virtually cyclic,

$$|U^3| > (\alpha |U|)^2 \text{ and in general } |U^n| > (\alpha |U|)^{\lfloor (n+1)/2 \rfloor}.$$