On mixed motivic sheaves and weights for them

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Weight structures and applications

Weight structures: "cousins" of t-structures (on a triangulated category $\underline{C}$); "two halves".

For $\underline{C} = K^b(\underline{B})$ or $= K(\underline{B})$: $\underline{C}_{w\le 0}$ = complexes, homotopic to ones concentrated in degrees ≥ 0; $\underline{C}_{w\ge 0}$ = complexes, $\backsim$ in degrees ≤ 0.

Weight structures give:

- weight complexes (vastly generalizing that of [GiS96]);
- $\underline{C}$-functorial weight filtrations and and spectral sequences (generalising coniveau and Deligne's weight ones);
- virtual t-truncations (that cut (co)homology into "pure" pieces), adjacent t-structures;
- other new (co)homological functors;
- $K_i(\underline{C})$ for $i \le 0$.

The heart $\underline{Hw} \subset \underline{C}$:

Obj $= \underline{C}_{w=0} = \underline{C}_{w\geq 0} \cap \underline{C}_{w\leq 0}$.

$\underline{Hw}$ is additive; *negative*: $\forall M, N \in \underline{C}_{w=0}$, $i > 0 \implies \underline{C}(M, N[i]) = \{0\}$ ($\underline{C}_{w=0} \perp \underline{C}_{w=i} = \underline{C}_{w=0}[i]$). Now,

Constructing weight structures is easy!

$\forall$ negative $\underline{B} \subset \underline{C}$ $\exists$ a *bounded* w on a subcategory of $\underline{C}$, with $\underline{B} \subset \underline{Hw}$.

Also some results for "big" $\underline{C}$ (compactly and well generated).

$S \in SH^c \subset SH$ gives $w^c_{sph} \subset w_{sph}$; $\underline{Hw} \cong$ (finitely generated) **free** abelian groups.

t_{sph} comes from cellular towers & calculates singular homology.

Definitions

B is Kar-closed in $\underline{C}$ if contains $\underline{C}$-retracts of its objects.

Definition 1 (B; D. Pauksztello)**.** (i) $\underline{C}_{w\ge 0}, \underline{C}_{w\le 0}$ are Karoubi-closed in $\underline{C}$.

(ii) $\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1], \underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.

(iii) **Orthogonality**. $\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.

(iv) **Decompositions**. $\forall M \in \operatorname{Obj} \underline{C}$ $\exists A \to M \to B$: $A \in \underline{C}_{w\le 0}, B \in \underline{C}_{w\ge 0}[1]$.

Similar to t-structures; yet different!

Shifted weight decompositions

$\forall M \in \operatorname{Obj} \underline{C}$, $m \in \mathbb{Z}$:

$w_{\le m}M \to M \to w_{\ge m+1}M \to (w_{\le m}M)[1]$

with $w_{\le m} \in \underline{C}_{w\le m} = \underline{C}_{w\le 0}[m]$ and $w_{\ge m+1}M \in \underline{C}_{w\ge m+1} = \underline{C}_{w\ge 0}[m+1]$.

Only weakly functorial.

Two types of w for (co)motivic categories: Chow and Gersten ones.

Chow weight structures on motives

$w_{Chow}(\operatorname{Spec} k)$ corresponding to $\mathrm{Chow}(k) \subset DM_{gm}(k)$ (with $\mathbb{Z}[\frac{1}{p}]$-linear coefficients). Extends to $\mathrm{DM}(k)$ and $DM_c(S) \subset \mathrm{DM}(S)$.

This gives:

- A conservative(!) exact functor $t : DM_{gm} \to K^b(\mathrm{Chow})$; $M \mapsto (M^i)$;
- $\forall\ H : {DM_{gm}}^{op} \to \underline{A}$ (abelian category), $i \in \mathbb{Z}$, a subfunctor $W_i H : M \mapsto \mathrm{Im}(H(w_{\geq i} M) \to H(M))$.
- Weight spectral sequence for a cohomological H: $E_1^{pq} = H^q(M^{-p}) \implies H^{p+q}(M)$; DM_{gm}-functorial starting from E_2 (which is determined by $t(M)$).
- $K_i(DM_{gm}) \cong K_i(\mathrm{Chow})$ for $i \leq 0$.
- Pure functors, adjacent t_{Chow} on DM.

An application to conservativity

If $\operatorname{char} k = 0$ then $\Omega'_{DR} = \Omega_{DR}^{t_{hom} \le 0}$ is a (h.s.) ring object in (k-linear) $\mathrm{DM}(k)$.

$$\begin{array}{ccccc}
\mathrm{Chow} & \to & DM_{gm} & \xrightarrow{t} & K^b(\mathrm{Chow}) \\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 \\
\mathrm{Mot}_{alg} & \to & \Omega'_{DR} - \mathrm{Mod}_c & \xrightarrow{t'} & K^b(\mathrm{Mot}_{alg}) \\
& & \downarrow H^*_{DR} & & \\
& & k - \mathrm{vect}^{\mathbb{Z}} & &
\end{array}$$

The first square from [BOg94, Corollary 7.4]; t and t' are weight complexes.

Since p_1 is conservative (by [Voe95]) and full, p_3 and p_2 are conservative.

If H^*_{DR} conservative (J. Ayoub), then $H^*_{DR} \circ p_2$ also is!

One can also apply weight spectral sequences here.

What if H^*_{DR} is conservative?

1) Then RH_{et} and vanishing cycles from $DM_c(S)$ are conservative for $S/\operatorname{Spec}\mathbb{Q}$.

2) $w_{Chow}(k)$ is a $\otimes$-structure; hence t (along with H^*_{DR}!) are $\otimes$-functors.
Thus conservativity of $H^*_{DR} \implies$ "Nori motives" in DM_{gm} are Kimura-finite i.e. for an affine A/k and its generic hyperplane section Z both
$M_{gm}(Z \to A)$ and $t(M_{gm}(Z \to A))$ are finite-dimensional (either higher exterior or higher symmetric powers $= 0$).

Hence numerical motives are Kimura-finite (can one do better??!);
motivic zetas are rational.

On relative motives and cross functors (Voevodsky, Ayoub, Cisinski, Deglise).

B = Noetherian separated excellent of finite Krull dimension;

B-schemes = separated f.t. $/B$,

$\mathcal{D}$: from B-schemes into (tensor) triangulated categories.

$\mathbf{1}_X$ = tensor unit of $\mathcal{D}(X)$ for a X/B.

For any B-morphism $f : X \to Y$ adjoint $f^* : \mathcal{D}(Y) \leftrightarrows \mathcal{D}(X) : f_*$ and $f_! : \mathcal{D}(X) \leftrightarrows \mathcal{D}(Y) : f^!$.

"Axioms": $f_* = f_!$ if f is proper; smooth and proper base change; gluing, etc.

For any $X/B : p_*(\mathbf{1}_{\mathbb{P}^1(X)}) \cong \mathbf{1}_X \bigoplus \mathbf{1}_X\langle -1\rangle$; $-\langle n\rangle = - \otimes (\langle -1_X\rangle)^{-n} = -(n)[2n]$.

When w_{Chow} exists

The *Chow-negativity* axiom required:

$\mathbf{1}(X) \perp \mathbf{1}(X)\langle i\rangle[n]$ $\forall i \in \mathbb{Z},\ n > 0$, $\forall$ regular X/B (or $X = \operatorname{Spec} K$).

Examples

(i) $\mathcal{D}(-) = \mathrm{DM}(-, R) =$ modules over $EM_R \in SH(-)$ (Voevodsky motives):
a) for "any" B and $R = \mathbb{Q}$;
b) for $1/p \in R$ and $\operatorname{char} B = p$
(we set "$1/0 = 1$");

(ii) "Cobordism-modules" (setting b));

(iii) K-motives ($KH[\mathcal{S}^{-1}]$-ones) $\forall$ B:
$\mathcal{S} = \{$primes not invertible on $B\}$,
$\mathcal{DK}(X) = \mathrm{KGl}'_X[\mathcal{S}^{-1}] - \mathrm{Mod}$.

Properties of "weights"

Similar to [BBD82, §5.1] and [Hub97].

$\mathbf{1}_X \in \mathcal{D}(X)_{w_{Chow}=0}$ if X is regular; f^* and $f_!$ "respect $w_{Chow} \leq 0$"; f_* and $f^!$ "respect $w_{Chow} \geq 0$".

$\mathcal{D}_{w_{Chow}(X) \leq 0}$ is "generated" by $f_!(\mathbf{1}_Y)\langle i \rangle$ for B-morphisms $f : Y \to X$, $i \in \mathbb{Z}$; $\mathcal{D}_{w_{Chow} \geq 0}$ is "generated" by $f_*(\mathbf{1}_Y)\langle i \rangle$ for regular Y.

$\underline{Hw}$ contains/ generated by $f_*(\mathbf{1}_Y)\langle i \rangle$ for regular Y and proper f: $\mathcal{D}$-Chow motives.

$\mathcal{D}$-Chow weight structures "glue"; can be detected pointwisely.

$\forall$ regular proper R/U, open $j : U \to X$

$\mathrm{Im}(K^*(R_X) \to K^*(R)) = W_0 H^*(j_!(u_! \mathbf{1}_R))$

$\forall$ "proper regular model" R_X/X

and $H = \mathcal{DK}(X)(-, \mathbf{1}_X)$:

"integral part of K-theory" (a-la Scholl).

Transversality with (motivic) t-structures

$\forall t$ on $\underline{C}$ gives: (i) abelian $\underline{Ht} \subset \underline{C}$ and (ii) t-homology $H : \underline{C} \to \underline{Ht}$.

$W^i H$ $\forall M \in \underline{C}^{t=0}$: "just a filtration"; "better" if all w.s.s. for H E_2-degenerate.

Transversality assumption:
$\forall M \in \underline{C}^{t=0},\ i \in \mathbb{Z}\ \exists\ w_{\leq i} M \in \underline{C}^{t=0}$.

Examples: $MHM(-)$ (and so, graded polarizable Hodge complexes), 1-motives, Artin-Tate motives/ #-fields.
For motives implies: pure motives $\mathrm{PM}_i(S) = \mathrm{Obj}\,\mathrm{MM} \cap \mathrm{DM}_{\mathbb{Q},c}(S)_{w_{Chow}=i}$ are semisimple (abelian); $\exists$ Chow-Künneth decompositions: $\mathrm{DM}_c(S)_{w_{Chow}=0} = \bigoplus \mathrm{PM}_i(S)[-i]$

"Standard" motivic conjectures (incl. Murre's ones) over universal domains $\Longrightarrow$ "nice weights" on $\mathrm{MM}(S) \subset \mathrm{DM}_{\mathbb{Q}}(S)$ $\forall$ variety S (and more generally).
D.T.: $\mathrm{PM}(S) = \bigoplus_{s \in S} j_{s!*}\,\mathrm{PM}(s)[\dim(\overline{s})]$.

Thank you for your attention!

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