Towards Embedding Theorem: $\mathrm{Aut}(\mathcal{D}_{\alpha e})$ embeds into $\mathrm{Aut}(\mathcal{D}_{\alpha})$

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Gödel's constructible hierarchy and admissibility [2]

Definition (Gödel's constructible hierarchy)

Defined inductively using the *first-order* definability:
$L_0 := \emptyset$, $L_{\gamma+1} := \mathrm{Def}(L_\gamma)$,
$L_\delta := \bigcup_{\gamma<\delta} L_\gamma$ if δ is a limit ordinal,
$L := \bigcup_{\gamma \in \mathtt{Ord}} L_\gamma$.

Definition (Admissible ordinal)

α is (Σ_1-)*admissible* iff $\forall f \in \Sigma_1(L_\alpha) \forall K \in L_\alpha . f[K] \in L_\alpha$ iff $L_\alpha \models \mathrm{KP}$.

Examples of admissible ordinals [2] [8]

- ω_1^{CK} - Church-Kleene ω_1, the first non-computable ordinal
- every stable ordinal α (i.e. $L_\alpha \prec_{\Sigma_1} L$), e.g. δ_2^1 - the least ordinal which is not an order type of a Δ_2^1 subset of $\mathbb{N}$, 1st stable ordinal
- every infinite cardinal in a transitive model of ZF

α-Computability Theory - basic concepts [2]

Definition

$K \subseteq \alpha$ is *α-finite* iff $K \in L_\alpha$.

Definition

- $f : \alpha \to \alpha$ is *α-computable* iff $f \in \Sigma_1(L_\alpha)$,
- A is α-computable iff $A \in \Delta_1(L_\alpha)$,
- A is α-computably enumerable (α-c.e.) iff $A \in \Sigma_1(L_\alpha)$.

Theorem (Koepke, Seyfferth 2009 [4])

Let an α-machine be a Turing machine with time α and tape length α.

- $f : \alpha \to \alpha$ is α-computable iff f is α-machine computable,
- A is α-computable iff A is α-machine computable,
- A is α-c.e. iff A is α-machine c.e.

α-Computability Theory - intuitions with higher definability

Theorem (Correspondence with second-order definability) [2] [5]

Let $n : \omega_1^{CK} \to \mathcal{O}$ take an ordinal to its computable notation in $\mathcal{O} \subseteq \omega$.

i) $\forall A \subseteq \omega[A \in \mathrm{HYP} \iff A \in \Delta_1^1 \iff A \in L_{\omega_1^{CK}}]$

ii) $\forall A \subseteq \omega_1^{CK}[A \in L_{\omega_1^{CK}} \iff n[A] \in \Delta_1^1]$

iii) $\forall A \subseteq \omega_1^{CK}[A \in \Sigma_1^0(L_{\omega_1^{CK}}) \iff n[A] \in \Pi_1^1]$

iv) $\forall A \subseteq \omega[A \in L_{\delta_2^1} \iff A \in \Delta_2^1]$

v) $\forall A \subseteq \omega[A \in \Sigma_1^0(L_{\delta_2^1}) \impliedby A \in \Sigma_2^1]$

Enumeration reducibilities

Proposition [2]

- There is an α-computable bijection $j : \alpha \to L_\alpha$.
- There is an α-computable bijection $p : \alpha \to \alpha \times \alpha$.

Hence we can index α-finite sets and reason about pairs.
Let K_γ denote $j(\gamma) \in L_\alpha$.

Definition (Enumeration reducibilities)

- $A \leq_{w\alpha e} B :\iff$
  $\exists \Phi \in \Sigma_1(L_\alpha) \forall x < \alpha [x \in A \iff \exists \delta < \alpha (\langle x, \delta \rangle \in \Phi \wedge K_\delta \subseteq B)]$
- $A \leq_{\alpha e} B :\iff$
  $\exists W \in \Sigma_1(L_\alpha) \forall \gamma < \alpha [K_\gamma \subseteq A \iff \exists \delta < \alpha (\langle \gamma, \delta \rangle \in W \wedge K_\delta \subseteq B)]$.

Weak reducibility $\leq_{w\alpha e}$ is not transitive in general.

Total reducibilities and degree structures

Definition (Total reducibilities)

- $A \leq_{w\alpha} B :\iff A \oplus \overline{A} \leq_{w\alpha e} B \oplus \overline{B}$.
- $A \leq_{\alpha} B :\iff A \oplus \overline{A} \leq_{\alpha e} B \oplus \overline{B}$.

Definition (Degree structures)

- $\mathcal{D}_{\alpha e} := 2^{\alpha} / \equiv_{\alpha e}$ are α-enumeration degrees,
- $\mathcal{D}_{\alpha} := 2^{\alpha} / \equiv_{\alpha}$ are α-degrees.

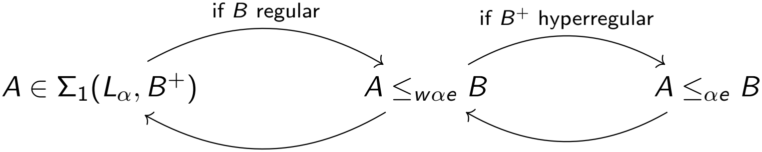
Let $\iota : \mathcal{D}_{\alpha} \hookrightarrow \mathcal{D}_{\alpha e}$ be an embedding induced by $A \mapsto A \oplus \overline{A}$.

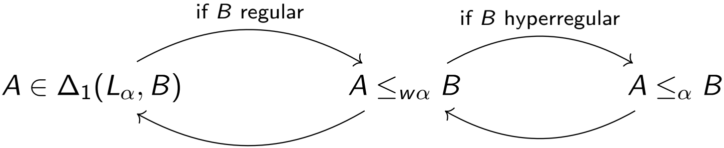
Definition (Total set and degree)

- A set $C \subseteq \alpha$ is *total* iff $C \equiv C \oplus \overline{C}$.
- A degree $\mathbf{d} \in \mathcal{D}_{\alpha e}$ is total iff $\exists C. C \oplus \overline{C} \in \mathbf{d}$.
- $\iota(\mathcal{D}_{\alpha})$ are *total* α-enumeration degrees.

Definition (Regularity and hyperregularity)

- A is *regular* iff $\forall \gamma < \alpha . A \cap \gamma \in L_\alpha$
- Let A be a set and A^+ its enumeration. Then $\mathcal{A} \in \{A, A^+\}$ is *hyperregular* iff $\forall f \in \Sigma_1(L_\alpha, \mathcal{A}) \forall K \in L_\alpha . f[K] \in L_\alpha$ iff α is $\Sigma_1(L_\alpha, \mathcal{A})$ admissible.





$$A \in L_\alpha \longleftrightarrow A \in \Delta_1(L_\alpha) \wedge \exists \delta < \alpha . A \subseteq \delta$$

Embedding Conjecture

Embedding Conjecture

$\exists \eta : \mathrm{Aut}(\langle \mathcal{D}_{\alpha e}, \leq \rangle) \hookrightarrow \mathrm{Aut}(\langle \mathcal{D}_{\alpha}, \leq \rangle)$.

Proof

Follows from the 3 statements:

1. $\mathcal{D}_\alpha$ degrees are embeddable in $\mathcal{D}_{\alpha e}$, i.e. $\exists \iota : \mathcal{D}_\alpha \hookrightarrow \mathcal{D}_{\alpha e}, A \mapsto A \oplus \overline{A}$
2. $\mathcal{D}_\alpha$ are an automorphism base for $\mathcal{D}_{\alpha e}$,
i.e. $\forall f \in \mathrm{Aut}(\mathcal{D}_{\alpha e})\ [f_{|\iota(\mathcal{D}_\alpha)} = 1_{\iota(\mathcal{D}_\alpha)} \implies f = 1_{\mathcal{D}_{\alpha e}}]$
Note (2) is implied by the conjecture generalizing a theorem of Selman [6]:
$\forall X \subseteq \alpha [B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}] \iff A \leq_{\alpha e} B$.
3. $\mathcal{D}_\alpha$ are definable in $\mathcal{D}_{\alpha e}$.

Then $\eta(f) := \iota^{-1} \circ f \circ \iota$ is the required embedding.

Definability of $\mathcal{D}_\alpha$ in $\mathcal{D}_{\alpha e}$

Definition (α-semicomputable set)

$A \subseteq \alpha$ is α-*semicomputable* iff $\exists s_A : \alpha \times \alpha \to \alpha \in \Sigma_1(L_\alpha) \forall x, y \in \alpha$:

- $s_A(x, y) \in \{x, y\}$
- $x \in A \vee y \in A \implies s_A(x, y) \in A$.

Definition (Kalimullin pair)

Kalimullin pair: $\mathcal{K}(A, B) :\iff \exists W \in \Sigma_1(L_\alpha)[A \times B \subseteq W \wedge \overline{A} \times \overline{B} \subseteq \overline{W}]$,
Maximal: $\mathcal{K}_{\max}(A, B) :\iff \mathcal{K}(A, B) \wedge$
$\forall C, D[A \leq_{\alpha e} C \wedge B \leq_{\alpha e} D \wedge \mathcal{K}(C, D) \implies A \equiv_{\alpha e} C \wedge B \equiv_{\alpha e} D]$,
Nontrivial: $\mathcal{K}_{\mathrm{nt}}(A, B) :\iff \mathcal{K}(A, B) \wedge A \notin \Sigma_1(L_\alpha) \wedge B \notin \Sigma_1(L_\alpha)$.

Conjecture

$\iota(\mathcal{D}_\alpha)$ are definable in $\mathcal{D}_{\alpha e}$:
$\forall c \in \mathcal{D}_{\alpha e}[c \text{ total iff } c = 0 \vee \exists a, b \in \mathcal{D}_{\alpha e}[(c = a \vee b) \wedge \mathcal{K}_{\max}(a, b)]]$

Conjecture

$\iota(\mathcal{D}_\alpha)$ are definable in $\mathcal{D}_{\alpha e}$:
$\forall c \in \mathcal{D}_{\alpha e}[c \text{ total iff } c = 0 \vee \exists a, b \in \mathcal{D}_{\alpha e}((c = a \vee b) \wedge \mathcal{K}_{\text{max}}(a, b))]$

Proof.

The conjecture follows from the 3 statements:
1. A Kalimullin pair is definable in $\mathcal{D}_{\alpha e}$,
$\mathcal{K}(a, b) :\iff \forall x \in \mathcal{D}_{\alpha e}.(a \vee x) \wedge (b \vee x) = x$.
2. $\forall c \in \iota(\mathcal{D}_\alpha) - \{0\} \exists a, b \in \mathcal{D}_{\alpha e}[(c = a \vee b) \wedge \mathcal{K}_{max}(a, b)]$,
2.1. $\forall c \in \iota(\mathcal{D}_\alpha) - \{0\} \exists C$ α-semicomputable st
$C \oplus \overline{C} \in c \wedge C \notin \Sigma_1(L_\alpha) \wedge C \notin \Pi_1(L_\alpha)]$ (Uses Shore's Splitting Theorem),
2.2. C is α-semicomputable $\wedge C \notin \Sigma_1(L_\alpha) \wedge C \notin \Pi_1(L_\alpha)$
$\implies \mathcal{K}_{max}(C, \overline{C})$,
3. $\forall a, b \in \mathcal{D}_{\alpha e}[\mathcal{K}_{max}(a, b) \implies \exists$ α-semicomputable C st $C \in a \wedge \overline{C} \in b]$,
thus $a \vee b$ contains $C \oplus \overline{C}$ and hence is total.

□

Kalimullin pair definability in $\mathcal{D}_{\alpha e}$

Definition (Projectum α^*)

$\alpha^* := \min\{\gamma < \alpha : \exists$ partial surjection $p_1 : \gamma \rightharpoonup \alpha \in \Sigma_1(L_\alpha)\}$

Using $p_1 : \alpha^* \to \alpha$ we can index all α-c.e. sets with α^*. Hence in priority arguments sufficient to satisfy only α^*-many arguments in α^* steps.

Theorem

Let $\alpha^* = \omega$ or $A \oplus B \oplus \emptyset'$ be hyperregular ($\emptyset'$ is an α-jump of $\emptyset$). Then:
$\mathcal{K}(A, B) :\iff \exists W \in \Sigma_1(L_\alpha)[A \times B \subseteq W \wedge \overline{A} \times \overline{B} \subseteq \overline{W}] \iff$
$\forall X \subseteq \alpha . \deg_{\alpha e}(X) = \deg_{\alpha e}(A \oplus X) \wedge \deg_{\alpha e}(B \oplus X)$

Every subset of an infinite regular cardinal is hyperregular. Thus:

Theorem (Kalimullin pair definability)

Let $\alpha^* = \omega$ or let α be an infinite regular cardinal. Then:
$\forall a, b \in \mathcal{D}_{\alpha e}[\mathcal{K}(a, b) \iff \forall x \in \mathcal{D}_{\alpha e} . x = (a \vee x) \wedge (b \vee x)]$.

Existence of α-semicomputable C & Labelling algorithm

Theorem

$\mathcal{K}_{nt}(A, B) \implies \exists C$ *α-semicomputable st* $A \leq_{\alpha e} C$ *and* $B \leq_{\alpha e} \overline{C}$.

Given a set $C \subseteq Q_\alpha = \{\sigma \in 2^{<\alpha} :\}$ and a labelling function $q_s : \alpha_A \coprod \alpha_B \rightarrowtail Q_\alpha$, define
$A_C := \{a \in \alpha_A : \exists s < \alpha . q_s(a) \in C\}$
$B_C := \{b \in \alpha_B : \exists s < \alpha . q_s(b) \in \overline{C}\}$.
The aim of the algorithm is to label α-rational numbers by the ordinals from α_A and α_B st there exists a cut C on the α-rational line (thus C is α-semicomputable) st $A_C = A$ and $B_C = B$. This implies the theorem.
Hence if $(a, b) \in A \times B$, then try to satisfy $q_s(a) < q_s(b)$,
if $(a, b) \in \overline{A} \times \overline{B}$, then try to satisfy $q_s(b) < q_s(a)$.
As $\mathcal{K}_{nt}(A, B)$, so $\exists W \in \Sigma_1(L_\alpha)[A \times B \subseteq W \wedge \overline{A} \times \overline{B} \subseteq W]$. Thus use W to try to satisfy these conditions.

Labelling algorithm - outline

1. Initially label the α-rationals on the left by the labels from α_B, label the α-rationals on the right by the labels from α_A.
2. Enumerate pairs from W,
3. Run a strategy (a, b) for $(a, b) \in \alpha_A \times \alpha_B$ according to its priority: gradually move a label $a \in \alpha_A$ towards left and the label from $b \in \alpha_B$ towards right trying to satisfy the conditions
   $(a, b) \in A \times B \implies q_s(a) < q_s(b)$ and
   $(a, b) \in \overline{A} \times \overline{B} \implies q_s(b) < q_s(a)$.
   Always require $q_s(a) < q_s(b) \implies (a, b) \in W$,
4. Every strategy will stop acting in less than α-steps, thus every pair of labels will eventually have a fixed position on the α-rational line.

Labelling algorithm

Note in some cases the labels cannot be ordered, e.g. when initially $q_s(b_2) < q_s(b_1) < q_s(a_1) < q_s(a_2)$, but next $(a_1, b_1) \searrow W_{s_1}$ and $(a_2, b_2) \searrow W_{s_2}$.

If $q_s(b) < q_s(a)$, but $(a, b) \in W$, then declare the interval $[q_s(a), q_s(b)]$ a dead zone (DZ) and prevent the pairs (a_i, b_i) with the strategies (a_i, b_i) of the lower priority than the strategy (a, b) moving the labels inside the dead zone. Let $q = \bigcup_{s<\alpha} q_s$. If $q(b) < q(a)$ and $(a, b) \in W$, then $[q(a), q(b)]$ is a permanent dead zone (PDZ).

Define the cut C and its complement D by

$C := \{\rho \in Q_\alpha : \exists b \notin B[\exists s < \alpha.\rho \leq q_s(b)\downarrow \text{ or } \{\rho, q(b)\} \subseteq \text{ a PDZ }]\}$.

$D := \{\rho \in Q_\alpha : \exists a \notin A[\exists s < \alpha.q_s(a)\downarrow \leq \rho \text{ or } \{\rho, q(a)\} \subseteq \text{ a PDZ }]\}$.

Then $C \cap D = \emptyset$, $A_C = A$ and $B_C = B$ as required.

Labelling algorithm in α-Computability Theory

- using Q_α instead of $\mathbb{Q}$,
- at a limit stage the labelling function is the union of the previous ones, i.e. for $q_\delta = \bigcup_{\gamma<\delta} q_\gamma$ for δ limit,
- two priority orderings for strategies: major by the order of the enumeration of the pair and the minor fixed (hence no clearing of the labels necessary),
- at stage $s < \alpha$ the strategy of the priority p will be allowed to label only the rationals with the binary representation of whose every nonzero substring (i.e. string containing at least one other symbol than 0) is of order type in $[(\Sigma_{\beta<s}\beta) + p, (\Sigma_{\beta<s+1}\beta) + p + \omega)]$. This is to guarantee that at every stage there would be enough space for new adjacent labels.
- termination of the strategy (a, b) is guaranted by the admissibility of α and the algorithm does not work for nonadmissible ordinals,
- break of the symmetry in proofs, e.g. in general it is not true that $\neg K \subseteq A \iff K \subseteq \overline{A}$ for $K \in L_\alpha$ unlike when $K = \{x\}$.

Summary of results

Theorem (Kalimullin pair definability for some α)

If $\alpha^* = \omega$ or α is an infinite regular cardinal, then $\mathcal{K}$-pair is definable in $\mathcal{D}_{\alpha e}$:
$\forall a, b[\mathcal{K}(a, b) \iff \forall x.x = (a \vee x) \wedge (b \vee x)]$

Theorem (Coditional definability of total degrees)

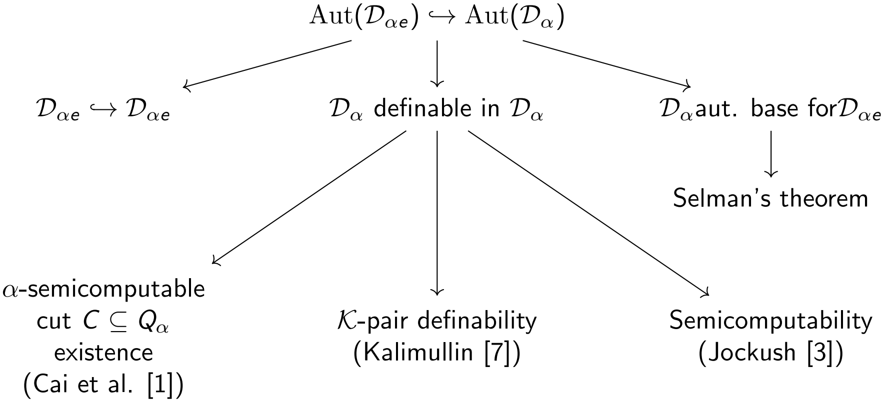
If $\mathcal{K}$-pair is definable in $\mathcal{D}_{\alpha e}$, then $\mathcal{D}_{\alpha}$ are definable in $\mathcal{D}_{\alpha e}$:
d total $\iff d = 0 \vee \exists a, b[\mathcal{K}_{\max}(a, b) \wedge (d = a \vee b)]$

Theorem (Conditional Embedding Theorem)

Assume that $\mathcal{K}$-pair is definable in $\mathcal{D}_{\alpha e}$ and Selman's theorem[a] holds for $\mathcal{D}_{\alpha e}$. Then $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_{\alpha})$.

[a] $\forall X \subseteq \alpha[B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}] \iff A \leq_{\alpha e} B$

Dependency graph



Discussion - oracle usage requires admissibility

- To prove the definability of $\mathcal{K}(A, B)$, we perform a construction using the oracle $A \oplus B \oplus \emptyset'$. Hence at the limit stages we need to use the hyperregularity of $A \oplus B \oplus \emptyset'$. Consequently, our theorem requires the assumption $\alpha^* = \omega$ or α is an infinite regular cardinal.
- On the other hand, given $\mathcal{K}_{\mathrm{nt}}(A, B)$ witnessed by W, with combinatorial methods we construct the α-computable labelling function $q : \alpha_A \cup \alpha_B \to Q_\alpha$ using $W \in \Sigma_1(L_\alpha)$ only. *After the construction* we define an α-semicomputable cut using q, A and B. But we do not use any hyperregularity of A or B. Hence the result holds for all admissible ordinals.
- One way to require less admissibility in our assumptions is to use less oracle power in a construction, then *later* show the existence of the desired incomputable objects by defining them using constructed α-c.e. objects and other input sets (e.g. A and B).

Discussion: Are there oracle-less proofs for open problems?

- Can we use combinatorial oracle-less constructions to solve open problems in α-Computability Theory established only for a certain classes of α using oracle construction (e.g. Post's problem, Kalimullin pair definability, Jump definability)?

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