YVES ANDRÉ: WHAT IS... A MOTIVIC GALOIS GROUP?

(Compiled from notes taken independently by Don Zagier and Herbert Gangl, quickly proofread by the speaker.)

We will be looking at fields K of characteristic 0. More precisely, think $\mathbb{Q}$ for the first part, and $\mathbb{C}(t)$ for the final part.

Recall that an element α which is algebraic over K has associated to it a Galois group $G(K(\alpha)^{\mathrm{gal}}/K)$ as a subgroup of some symmetric group $\mathcal{S}_n$. Writing $G_K = \mathrm{Gal}(\overline{K}/K)$, we have the Grothendieck-Galois correspondence (an equivalence of categories between finite G_K-sets and finite varieties over K)

$$\begin{aligned} G_K\text{–FinSets} \quad &\leftrightarrow \quad \mathrm{FinVar}/K \\ X(\overline{K}) \quad &\leftrightarrow \quad X = \mathrm{Spec} \prod_i L_i \end{aligned}$$

with $[L_i : K] < \infty$ for any i.

Grothendieck's motivic Galois theory now can be described as

(A) a generalization to several polynomials in several variables, that is to higher dimensional varieties $/K$
(passage (pro-)finite groups $\longrightarrow$ (pro-)linear groups $/\mathbb{Q}$), or
(B) keeping in mind that (for $K = \mathbb{Q}$, say) algebraic numbers are *periods*, as a generalization of Galois theory.

While (A) has been finally established in 2015, (B) still remains a dream and looks inaccessible—it is the holy grail of transcendence theory.

Note that while π (which is well-known to be a period) is known to be transcendental, Euler found that it is the 'root of a polynomial of infinite degree' as

$$\prod_{n\neq 0}\Big(1 - \frac{x}{n\pi}\Big) = \frac{\sin(x)}{x} \in \mathbb{Q}[[x]]$$

where the 'conjugates' of π are the elements of $\mathbb{Q}^\times\pi$, this suggests to consider the subgroup $\mathbb{G}_m$ of $\mathrm{Aut}\big(\mathbb{Q}[\pi]/\mathbb{Q}\big)$ as the Galois group attached to $\mathbb{Q}[\pi]$.

1. Pure motivic Galois groups

Grothendieck's idea arose from enumerative geometry (the study of configurations in projective spaces)

$$\sum n_i Z_i = Z \subset X\,, \qquad n_i \in \mathbb{Q}\,,$$

modulo numerical equivalence given by

$$Z \sim_{\mathrm{num}} 0 \quad \text{if} \quad Z\cdot Z' = 0 \quad \forall Z' \text{ of complementary dimension}\,.$$

Start with linearizing smooth projective varieties

$$X, Y \in \mathrm{SmProjVar}/K, \qquad \text{correspondences } Z \subset X{\times}Y, \quad \dim Z = \dim Y\,, \qquad \text{mod } \sim_{\mathrm{num}}\,.$$

Grothendieck conjectured, and Jannsen (in 1991) found a "miraculous" proof for the fact that the category $\mathcal{M}^+_{\mathrm{num}}$ of pure numerically effective motives is *abelian semisimple*, where $\mathcal{M}^+_{\mathrm{num}}$ is obtained from the above $\mathbb{Q}$-linear category by formally adding kernels of idempotents, thus a priori giving a pseudo-abelian category only.

The tensor product $\otimes$ arises from the cartesian product of varieties. The idea of linearization leads e.g. to a decomposition $[\mathcal{P}^1] = [pt] \oplus \mathbb{L}$ where $\mathbb{L}$ is the Lefschetz motive, which then is getting formally inverted, leading to the category of pure numerical motives

$$\mathcal{M}_{\text{num}} = \mathcal{M}^+_{\text{num}}\Big[\frac{1}{\mathbb{L}}\Big]$$

in which every object has a dual with respect to $\otimes$. $\mathcal{M}_{\text{num}}$ is again an abelian semisimple $\mathbb{Q}$-linear category.

How do we get a group out of $\mathcal{M}_{\text{num}}$? For the previous equivalence of categories

$$\begin{array}{ccc} \text{FinVar}/K & \stackrel{\varphi}{\to} & G_K\text{–FinSets} \\ X & \mapsto & X(\overline{K}) \end{array}$$

the Galois group G_K can be reconstructed as $\text{Aut}(\varphi)$ (the automorphism group of the functor φ) which we can also write $G_K = \varprojlim_{[L:K]<\infty} G(L)$.

Grothendieck's idea now was to linearize this construction, introducing *tannakian* categories: one considers (C, ω) where

$$C \stackrel{\omega}{\longrightarrow} \text{Vec}_F$$

where C is an F-linear abelian $\mathbb{Q}$-category in which every object has a dual and where ω is a faithful exact $\otimes$-functor into the category of finite dimensional vector spaces.

Now putting $G = \text{Aut}^{\otimes}\omega$ (i.e., the automorphism group of the fibre functor ω), the tannakian category property guarantees that C is equivalent to the category of (finite dimensional) representations of this group G, so we can write

$$C \stackrel{\text{eq}}{\simeq} \text{Rep}_F G \longmapsto \text{Vec}_F\,,$$

and any given object $M \in Ob(C)$ inherits an action of G via $\text{Aut}^{\otimes}\omega\ |_{\langle M\rangle_{\otimes}} = G(M) < GL(\omega(M))$, where we view $G(M)$ as a linear algebraic group.

Can one apply this to $C = \mathcal{M}_{\text{num}}$, with $\omega = H_B$ (Betti homology) and $K \subset \mathbb{C}$, say?

The *problem* here is that H_B is not defined on MM_{num} unless $\sim_{\text{num}}=\sim_{\text{hom}}$ (i.e. if numerical equivalence equals homological equivalence, one of Grothendieck's "standard conjectures"). Without this, things break down, and for a long time it seemed that one could not construct a theory of motives without proving the standard conjectures.

But this is no longer true. One can overcome this problem in a 'minimal way' (André 1996) by modifying $\mathcal{M}_{\text{num}}(K)$ as follows. For $X \in \text{SmProj}/K$ of dimension d one has the Lefschetz isomorphism(s)

$$H^i(X) \xrightarrow[\sim]{\eta^{d-i}} H^{2d-i}(X)$$

for any $i \leqslant d$ and any polarization $\eta \in H^2(X)$. Define an involution $*_L$ of $H(X)$ by the cup product with η^{d-i} if $i \leqslant d$ and by $(\eta^{d-i})^{-1}$ if $i > d$.

This is *not known to be algebraic* (in fact, $*_L$ is algebraic $\Leftrightarrow$ standard conjectures hold).

Then define $\mathcal{M}(K)$ by declaring the morphisms to be the algebraic correspondences modulo $\sim_{\text{hom}}$, and add formally the above $*_L$'s.
This is still tannakian and semisimple (the Hodge index theorem guarantees that a slight variant $*_H$ of $*_L$ gives *positive* intersection numbers), giving rise to André's unconditional theory of motives. This category is the minimal way to make Grothendieck's construction work, and it agrees with the latter if (and only if) the standard conjectures are true.

Now the **(pure) motivic Galois group** in this setting is the following pro-reductive algebraic group:

$$\begin{aligned} G^{\text{pure}}_{\text{mot},K} &= \text{Aut}^{\otimes} H_B \\ &= \varprojlim_{X \in \text{SmProjVar}/K} G^{\text{pure}}_{\text{mot},K}\big(\langle X\rangle_{\otimes}\big), \end{aligned}$$

where $\langle X\rangle_{\otimes} \subset \mathcal{M}$ (note that $G^{\text{pure}}_{\text{mot},K}\big(\langle X\rangle_{\otimes}\big) < GL(H(X))$).

Summary:

numerical equivalence (gives abelian category)
↑
homological equivalence (not known to give ab. cat.)
↓
homological equivalence and invert Lefschetz. (ab. cat.)

How can we compute such a motivic Galois group?

Note that it can already be very difficult to compute ordinary Galois groups, and here we follow the same strategy: first find an 'upper bound', then try to show equality. But now the Galois groups are linear algebraic (rather than finite) groups and so one needs some classical representation theory (weights, ...) together with invariant theory.

Example 1. $X = \operatorname{Spec} L$, $[L : K] < \infty$.
Then $G^{\text{pure}}_{\text{mot},K}(X) = \text{Gal}(L^{\text{gal}}/K)$.

Example 2. $X = \mathbb{L}$, $G(\mathbb{L}) = \mathbb{G}_m$.

Example 3. $X =$ elliptic curve$/K$, we get $h(X) = h(pt.) \oplus h^1(X) \oplus \mathbb{L}$,
noting that $\bigwedge^2 h^1(X) = \mathbb{L}$; and we get the reductive group

$$G(X) = G(h^1(X)) \subset GL(H^1(X)) = GL_2$$

(but we get several possibilities, see below). Because of $\bigwedge^2 h^1(X) = \mathbb{L}$ we have a determinant map $\det : G(X) \twoheadrightarrow \mathbb{G}_m$; the center of the kernel of det is compact.

One finds that there are exactly 3 possibilities:

(i) G = non-split Cartan subgroup of $GL_2(\mathbb{Q})$ (= $Re_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}\mathbb{G}_m$, via restriction from some imaginary quadratic field);
(ii) G = normalizer of a non-split Cartan subgroup (in which case the normalizer has order 2);
(iii) $G = GL_2$,

and this corresponds, after some work, to

(i) X has CM(=complex multiplication) over K (i.e. $(End_K X) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-d})$);
(ii) X has CM over some quadratic extension of K (i.e. $(End_{\overline{K}} X) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-d})$);
(iii) X has no CM over $\overline{K}$ (i.e. $(End_{\overline{K}} X) \otimes \mathbb{Q} = \mathbb{Q}$).

Example 4. Abelian varieties with CM$/K = \overline{K}$.
CM-type: $\tau : \text{Gal}(\mathbb{Q}^{\text{CM}}/\mathbb{Q}) \to \mathbb{Z}$, where $\tau(g) + \tau(\overline{g}) =$ const. The *Serre protorus* S is the protorus over $\mathbb{Q}$ with characters the CM types.

Theorem. (André '96)
The motivic Galois group attached to the tannakian subcategory $\mathcal{M}^{\text{ab CM}} \subset \mathcal{M}$ generated by CM abelian varieties is

$$G^{\text{ab CM}} = S\ .$$

Now for the **mixed** theory.

2. Mixed motivic Galois groups

Grothendieck predicted this case as well, more precisely he conjectured the existence of a category $\mathcal{MM}(K)$ with 3 properties

(1) "universal cohomology": the maps to different cohomology theories (now allow *all* varieties $/K$, not just smooth projective ones)

$$\mathrm{Var}_K \rightrightarrows \begin{matrix} H_B(+\,H_{dR}) \\ H_\ell \end{matrix}$$

should factor through $\mathcal{MM}(K)$;

(2) "geometric nature": the morphisms come from geometry (so while they maybe not be algebraic correspondences themselves, they are *constructed in terms of* algebraic correspondences);

(3) it should be a tannakian category, so one should have an equivalence of categories (the composed map being H_B, the Betti homology functor)

$$\mathcal{MM}(K) \xrightarrow{\sim} Rep_{\mathbb{Q}} G_{\mathrm{mot},K} \dashrightarrow \mathrm{Vec}_{\mathbb{C}}\,,$$

for some group $G_{\mathrm{mot},K}$, which then is called the **mixed motivic Galois group**.

For many years this was just science fiction; the work of Deligne made it *useful* fiction; then 2 years ago (in 2015) it became non-fiction thanks to work of Nori and Ayoub.

Moreover, the semisimple subcategory (the full subcategory of semisimple objects) $\mathcal{MM}_{ss}(K)$ coincides with $\mathcal{M}(K)$ = the pure motivic category as constructed by André (proved by Arapura).

Earlier work in this direction was done by Deligne, Jannsen, Huber and others using systems of realizations $\mathcal{H}$. We can say now that we have a functor

$$\mathcal{MM}(K) \longrightarrow \underbrace{\mathcal{H}^{\mathrm{m}}}_{\textit{essential image}} \subset \mathcal{H}\,.$$

Objects in $\mathcal{H}^{\mathrm{m}}$ are sometimes called absolute Hodge 'motives', albeit they are not [known to be(?)] 'motives' in the above sense since the morphisms $G^{\mathrm{pure}}_{\mathrm{mot},K} = (G_{\mathrm{mot},K})^{\mathrm{red}}$ are not [known to be(?)] geometric.

One has

$$G^{\mathrm{m}}_K \subset G_{\mathrm{mot},K}$$

and some people conjecture this to be an equality.

Key idea (Ayoub): construct a tannakian category starting from Voevodsky's triangulated category.

Nori's construction: he considered pairs of algebraic varieties $(X, Y), Y \subset X$, (which do not form a category yet as there are no compositions of morphisms) but a quiver where the arrows are given by all morphisms in all cohomology theories; then out of such a quiver one has a way to cook up a tannakian category which is universal with respect to abelian categories. The crucial (and hard) part is the $\otimes$ product.

Ayoub's contribution: a much more complicated new tannakian category which applies to a very general context; by using new adjoints, he constructs a Hopf algebra (or rather a differential graded Hopf algebra), for which things live in non-negative degrees, and hence at the end he truncates artificially (note that this step is empty if the Beilinson-Soulé conjecture holds). Subsequently two students of Ayoub (Choudhury, Gallauer) proved that the two theories [of Nori and Ayoub] are the *same* (this was rather non-trivial as the constructions are completely different.) For details, see André's talk in the Bourbaki seminar 2016.

[Aside: There are two completely distinct 'packets' of conjectures:

(i) pertaining to "motives": Beilinson-Soulé type conjectures, existence of motivic t-structure etc., which ensure e.g. that $\mathrm{Ext}^*_{\mathcal{MM}}$ give K-groups.

(ii) pertaining to "realizations": Conjectures about fullness of enriched realizations (such as the Hodge Conjecture, the Tate Conjecture, the conjecture that $\mathcal{MM} \cong \mathcal{H}^{\mathrm{m}}$ etc.)]

3. Links to periods

We have the following diagram

$$\mathcal{MM}(K) \longrightarrow \mathcal{H}^{\mathrm{m}} \longrightarrow \mathcal{T} \rightrightarrows \mathrm{Vec}_{\mathbb{Q}}$$

where $\mathcal{T} = \{(V, W, V_{\mathbb{C}} \simeq W_{\mathbb{C}})\}$ and two fiber functors ω_B ('take the W-part' in the right hand map) and ω_{dR} ('take the V-part' in the right hand map) for the right hand maps, respectively, leading to the compositions H_B and H_{dR}.

For this section, assume $K = \mathbb{Q}$.

We can define $\mathrm{Isom}^{\otimes}(H_{dR}, H_B) = \mathrm{Spec}\, P_{\mathrm{mot},\mathbb{Q}}$ (the true motivic period torsor—a torsor under $G_{\mathrm{mot},K}$), and its avatar for $\mathcal{H}^{\mathrm{m}}$ $\mathrm{Isom}^{\otimes}(H_{dR}\mid_{\mathcal{H}^{\mathrm{m}}}, H_B\mid_{\mathcal{H}^{\mathrm{m}}}) = \mathrm{Spec}\, P^{\mathrm{m}}$ agrees with the G^{m}_K-torsor $\mathrm{Spec}\, P^{\mathrm{m}}$ in Brown's talk earlier this week.

More precisely, one has

$$\mathcal{P}_{\mathrm{mot}} \twoheadrightarrow \mathcal{P}^{\mathrm{m}} \overset{\mathrm{per}}{\twoheadrightarrow} \mathcal{P} \subset \mathbb{C}$$

where $\mathcal{P}$ denotes polynomials in $1/\pi$ with coefficients in the ring of periods.

Grothendieck's period conjecture (=GPC) is that the composed map above, denoted Per, is injective (this constitutes a rather wild conjecture); the injectivity of per, is weaker but also completely conjectural.

If the strong conjecture holds, then there is a $G_{\mathrm{mot},K}$-action on periods;
if only per is injective, then the smaller group G^{m}_K acts.

Concrete example. Consider $\mathcal{M}^{\text{ab, cycl CM}} \subset \mathcal{M}$, generated by abelian varieties with cyclotomic CM. Restrict GPC-type conjecture to this smaller category.

Theorem. (non-trivial, cf. the penultimate chapter of André's book.) The GPC conjecture for this category is equivalent to the conjecture that all algebraic relations among Γ-values $\Gamma(r)$, $r \in \mathbb{Q}$, come from the standard functional equations of Γ (i.e. $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(x)\Gamma(1-x) = \pi/\sinh(\pi x)$ as well as the distribution relations).

Example: $\Gamma\left(\frac{1}{n}\right)$ is known to be transcendental for $n = 3, 4, 6$ but not known to be for $n = 5$.

GPC implies that $\mathcal{MM}(K) \longrightarrow \mathcal{T}$ (given by Betti cohomology, de Rham cohomology, together with the isomorphism of their complexifications) is full. This weaker statement, restricted to $\mathcal{M}^{\text{ab, cycl CM}}$, corresponds to just the *multiplicative* relations among $\{\Gamma(r), r \in \mathbb{Q}\}$ coming from functional equations of $\Gamma(x)$.

4. The functional case

Now we have $K = \mathbb{C}(t)$ instead of $\mathbb{Q}$. As K can no longer be embedded into $\mathbb{C}$, we work with H_{dR} rather than H_B. We have periods $\int_\Delta \omega$ where ω depends algebraically on t. Such functions satisfy a Picard–Fuchs differential equation (Fuchsian, so only regular singular points can occur). These are subject to a differential Galois theory, so we have that $\mathrm{Spec}\, \mathcal{P}_K$ is a torsor under a universal differential Galois group G (we get for free that the group is acting on periods).

Since the differential equations are Fuchsian, this G is some version of a monodromy group, $G = G_{\mathrm{mono},K}$. In fact, there is also a relationship of this with motives:

Theorem. (Ayoub, '05):

$$\begin{aligned} G_{\mathrm{mono},K} &= \ker\left(G^{dR}_{\mathrm{mot},K} \longrightarrow G_{\mathrm{mot},\mathbb{C}} \otimes K\right) \\ &= G^{\mathrm{rel}}_{\mathrm{mot},K}\,. \end{aligned}$$

The corresponding fact in the pure case had been shown by André ('96).

[This is a motivic avatar of Deligne's theorem of the fixed part in Hodge theory.]

This leads to a remarkable *presentation* of the ring of functional periods $\mathcal{P}_K$ by generators and relations.

Consider holomorphic functions on the closed polydisk $f \in \mathcal{O}(\overline{D}^n)((t))^{\mathrm{alg}}$, giving algebraic elements over $\mathbb{C}(z_1, \ldots, z_n, t)$, where $\overline{D}$ is given via the conditions $|z_i| \leqslant 1$ for $i = 1, \ldots, n$, and we can write $f = \sum_m f_m t^m$.

We can integrate, for arbitrary $n \geqslant 0$,

$$\int_{[0,1]^n} f = \sum_m \left(\int_{[0,1]^n} f_m \right) t^m \in \mathbb{C}((t)) .$$

(Think of the interval $[0, 1]$ being embedded into the closed unit disk.)

Altogether get a map $f \mapsto \int_\square f$, so $\int_\square : \bigcup_n \mathcal{O}(\overline{D}^n)((t))^{\mathrm{alg}} \to \mathbb{C}((t))$, and we have

Theorem. (Ayoub '15)

1) Any period which is meromorphic at 0 lies in the image *im* $\int_\square$ (maybe after enlarging the number of variables).
2) $\ker \int_\square$ is $\mathbb{C}$-spanned by two types

$$(i)\ \frac{\partial}{\partial z_i} - h\,|_{z_i=1} + h\,|_{z_i=0} \qquad \text{and} \qquad (ii)\ \Big(f - \int_\square f\Big) g$$

with $f, g, h \in \mathcal{O}(\overline{D}^n)((t))^{\mathrm{alg}}$. Here f does not depend on t, and f and g do not depend on any common z.