The isomorphism problem for polycyclic groups

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1 The isomorphism problem

The isomorphism problem considers two groups G and H and asks whether these are isomorphic; if so, then determine an explicit isomorphism. Dual to that is the computation of generators for $Aut(G)$.
In general, the isomorphism problem is undecidable, see Adian (1957) and Rabin (1958).

- If G is polycyclic, then $G/Fit(G)$ is abelian-by-finite.
- Two key-cases: abelian-by-finite and fg nilpotent.
- Eick (2016): practical method for abelian-by-finite polycyclic groups.
- Here: consider the fg nilpotent case in more detail.

- If G is fg nilpotent, then $T(G)$ exist and is finite.
- $G/T(G)$ is tf fg nilpotent - key case.
- Grunewald & Segal (1980) show that the isomorphism problem for tf fg nilpotent groups is decidable.
- Eick & Engel (2017) exhibit a practical method for small cases.

2 Grunewald & Segal

Recall the main ideas of this approach.

- Let C be the category of finite presentations of tf nilpotent groups with morphisms = isomorphism.
- Let $B = \cup B_n$ and B_n the category of subgroups of $Tr_1(n, \mathbb{Z})$ with morphisms = conjugation.
- Then there exists a faithful functor $\theta : C \to B$.
- Given G and H tf nilpotent groups:
  - determine $\theta(G)$ and $\theta(H)$
  - check that the dimensions n_G and n_H agree
  - check that $\theta(G)$ and $\theta(H)$ are conjugate in $GL(n_G, \mathbb{Z})$.
- Gien G tf nilpotent group:
  - determine $I = \theta(G)$
  - compute $N_{GL}(I)$ and $C_{GL}(I)$ as arithmetic groups
  - yields $Aut(G) = N_{GL}(I)/C_{GL}(I)$.

The construction of the functor is practical in very small cases only, see Lo & Ostheimer (1999). Given G, the image $\theta(G)$ is obtained as right regular representation of G acting on a quotient $\mathbb{Z}G/J$, where $J/I^{c+1} = tor(\mathbb{Z}G/I^{c+1})$ with I the augmentation ideal and c the class of G. The dimension of the image $\theta(G)$ is usually quite large.

Isomorphism testing now translates to finding a conjugating element for two unitriangular matrix groups in $GL(n, \mathbb{Z})$. This may be doable?

Dual to that, is it possible to compute generators for the normalizer of a unitriangular matrix group in $GL(n, \mathbb{Z})$? If so, then the arithmetic group part of the algorithm may perhaps be skipped.

3 Eick & Engel

Consider the main ideas of this approach. Each fg tf nilpotent group has a presentation on n generators $g_1, \ldots, g_n$ with relations

$$g_j g_i = g_i g_j v_{i,j} \text{ for } 1 \leq i < j \leq n$$

and

$$v_{i,j} = g_{j+1}^{t_{i,j,j+1}} \cdots g_n^{t_{i,j,n}}.$$

Hence this presentation is determined by the exponents $t = (t_{i,j,k} \mid 1 \leq i < j < k \leq n)$. Call this presentation $P(t)$. For a tf fg nilpotent group G let

$$T(G) = \{t \in \mathbb{Z}^{\binom{n}{3}} \mid G \cong P(t)\}.$$

Idea: define a *canonical element* in $T(G)$ and an algorithm to determine it.
We determined such elements for all groups of Hirsch length at most 5 and algorithms to determine these. The algorithms are highly practical.