Isomorphism problem

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Oct 2018

Isolator series

Definition

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- Let $G = \gamma_1(G) > \ldots > \gamma_c(G) > \{1\}$ be its lower central series.
- Define $I_i(G)/\gamma_i(G)$ as the torsion subgroup of $G/\gamma_i(G)$.
- Then
$$G = I_1(G) > \ldots > I_c(G) > \{1\}$$
is the *isolator series* of G.

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Note

- The type is an isomorphism invariant.
- $d_1 + \ldots + d_c$ is the Hirsch length of G.

List of types

Possible types by Hirsch length h and class c

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
$c=1$	(1)	(2)	(3)	(4)	(5)
$c=2$			(2,1)	(3,1)	(4,1) (3,2)
$c=3$				(2,1,1)	(3,1,1) (2,1,2)
$c=4$					(2,1,1,1)

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	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$
$c = 1$	(1)	(2)	(3)	(4)	(5)
$c = 2$			(2,1)	(3,1)	(4,1) (3,2)
$c = 3$				(2,1,1)	(3,1,1) (2,1,2)
$c = 4$					(2,1,1,1)

Note

- $c = 1$: these groups are free abelian.
- $c = 2$: Grunewald & Scharlau (1979) solved isomorphism problem.
- $c = 3$ and $c = 4$: our aim!

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- Determine multiplication polynomials (Hall polynomials) for the groups of this type.
- Use them to translate the isomorphism problem or the automorphism problem to a set of polynomial equations.
- Solve the set of polynomial equations. (Requires a variety of techniques.)

Type $(3, 1, 1)$

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Let G be of type $(3, 1, 1)$.

- Isolator series $G = I_1(G) > I_2(G) > I_3(G) > I_4(G) = \{1\}$.
- Let $C = C_G(I_2(G))$.
- Then
  $$G > C > I_2(G) > I_3(G) > I_4(G) = \{1\}$$
  is a fully invariant central series with free abelian quotients of ranks $1, 2, 1, 1$.

Presenations for type $(3, 1, 1)$

Presentation

G has a presentation $G(t)$ with generators $g_1, \ldots, g_5$ and relations

$$
\begin{aligned}
[g_2, g_1] &= g_4^{t_{124}} g_5^{t_{125}}, \\
[g_3, g_1] &= g_4^{t_{134}} g_5^{t_{135}}, \\
[g_4, g_1] &= g_5^{t_{145}}, \\
[g_3, g_2] &= g_5^{t_{235}}, \\
&\quad \text{all other commutators trivial.}
\end{aligned}
$$

Canonical forms for type $(3, 1, 1)$

Theorem (E. & Engel (2016))

Let $t = (t_{124}, t_{134}, t_{235}, t_{145}, t_{125}, t_{135})$ and $G = G(t)$ of type $(3, 1, 1)$.

- Let $a = \gcd(t_{124}, t_{134})$.
- Let $L \in GL(2, \mathbb{Z})$ with $(t_{124}, t_{134})L = (a, 0)$.
- Let $l = (t_{125}, t_{135})L$.
- Let $c = \min\{\pm l_2 \bmod \gcd(t_{145}, t_{235})\}$.
- Let $b = \min\{\pm l_1 \bmod \gcd(l_2, t_{124}, t_{134}, t_{145}, t_{235})\}$.
- Then a canonical element in $T(G)$ is

$$(a, 0, |t_{235}|, |t_{145}|, b, c).$$

Example application

Example

Consider presentation

$$
\begin{aligned}
[g_2, g_1] &= g_4^{9} g_5^{23}, \\
[g_3, g_1] &= g_4^{18} g_5^{3}, \\
[g_4, g_1] &= g_5^{-7}, \\
[g_3, g_2] &= g_5^{21}, \\
&\text{all other commutators trivial.}
\end{aligned}
$$

Example application

Example

Green: make positive

$$[g_2, g_1] = g_4^{9} g_5^{23},$$
$$[g_3, g_1] = g_4^{18} g_5^{3},$$
$$[g_4, g_1] = g_5^{7},$$
$$[g_3, g_2] = g_5^{21},$$

all other commutators trivial.

Example application

Example

Red: invoke $\gcd(9, 18) = 9$ (modifiy blue accordingly)

$$[g_2, g_1] = g_4^{9} g_5^{23},$$
$$[g_3, g_1] = g_4^{0} g_5^{-41},$$
$$[g_4, g_1] = g_5^{7},$$
$$[g_3, g_2] = g_5^{21},$$
$$\text{all other commutators trivial.}$$

Example application

Example

Blue: reduce modulo gcd's.

$$
\begin{aligned}
[g_2, g_1] &= g_4^{9} g_5^{0}, \\
[g_3, g_1] &= g_4^{0} g_5^{1}, \\
[g_4, g_1] &= g_5^{7}, \\
[g_3, g_2] &= g_5^{21}, \\
& \text{all other commutators trivial.}
\end{aligned}
$$

Example application

Example

Result

$$[g_2, g_1] = g_4^{9},$$
$$[g_3, g_1] = g_5^{1},$$
$$[g_4, g_1] = g_5^{7},$$
$$[g_3, g_2] = g_5^{21},$$

all other commutators trivial.

Automorphisms for type $(3, 1, 1)$

Each automorphism has the form

$$\begin{aligned}
g_1 &\mapsto g_1^{m_{11}} g_2^{m_{12}} g_3^{m_{13}} g_4^{m_{14}} g_5^{m_{15}}, \\
g_2 &\mapsto g_2^{m_{22}} g_3^{m_{23}} g_4^{m_{24}} g_5^{m_{25}}, \\
g_3 &\mapsto g_2^{m_{32}} g_3^{m_{33}} g_4^{m_{34}} g_5^{m_{35}}, \\
g_4 &\mapsto g_4^{m_{44}} g_5^{m_{45}}, \\
g_5 &\mapsto g_5^{m_{45}}.
\end{aligned}$$

Automorphisms for type $(3, 1, 1)$

And thus corresponds to an integral matrix M

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ 0 & m_{22} & m_{23} & m_{24} & m_{25} \\ 0 & m_{32} & m_{33} & m_{34} & m_{35} \\ 0 & 0 & 0 & m_{44} & m_{45} \\ 0 & 0 & 0 & 0 & m_{55} \end{pmatrix}$$

Automorphisms for type $(3, 1, 1)$

Theorem (E. & Engel (2016))

Let $G(T)$ be a canonical form of type $(3, 1, 1)$. Then M is an automorphism of $G(T)$ if and only if $M \in GL(5, \mathbb{Z})$ and has the form

$$\begin{pmatrix} f & m_{12} & m_{13} & m_{14} & m_{15} \\ 0 & s & m_{23} & m_{24} & m_{25} \\ 0 & 0 & e & m_{34} & m_{35} \\ 0 & 0 & 0 & sf & m_{45} \\ 0 & 0 & 0 & 0 & s \end{pmatrix}$$

with $e, f, s \in \{\pm 1\}$ with $e = 1$ if $T_{235} \neq 0$ and

- $m_{34}T_{145} + fm_{12}T_{235} = (sf - e)T_{135}$, and
- $m_{45}T_{124} - fm_{23}T_{135} - (m_{12}m_{23} - sm_{13})T_{235} - fm_{24}T_{145} = s(f - 1)/2(2T_{125} - T_{145}T_{124})$.

Automorphisms for type $(3,1,1)$

Note

$$\varphi : \mathrm{Aut}(G) \to \mathrm{GL}(3,\mathbb{Z}) : M \to \begin{pmatrix} f & m_{12} & m_{13} \\ 0 & s & m_{23} \\ 0 & 0 & e \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z})$$

is a group homomorphism with nilpotent kernel.

Automorphisms for type $(3,1,1)$

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$$\varphi : \mathrm{Aut}(G) \to \mathrm{GL}(3,\mathbb{Z}) : M \to \begin{pmatrix} f & m_{12} & m_{13} \\ 0 & s & m_{23} \\ 0 & 0 & e \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z})$$

is a group homomorphism with nilpotent kernel.

Corollary

Let G be of type $(3,1,1)$. Then $\mathrm{Aut}(G)$ is polycyclic.

Example application

Consider again the example

- $m_{34} = ((sf - 1) - 7fm_{12})/21.$
- $m_{45} = (fm_{23} + 7(m_{12}m_{23} - sm_{13}) + 21fm_{24})/9 + s(1 - f)21/2.$

Example application

The image of φ has the form

$$\{\begin{pmatrix} f & 3x & z \\ 0 & f & 3y+z \\ 0 & 0 & 1 \end{pmatrix} \mid f = \pm 1, x, y, z \in \mathbb{Z}\}.$$

Example application

The image of φ has the form

$$\{\begin{pmatrix} f & 3x & z \\ 0 & f & 3y+z \\ 0 & 0 & 1 \end{pmatrix} \mid f = \pm 1, x, y, z \in \mathbb{Z}\}.$$

Since

- $sf = -1$ is impossible, hence $s = f$ and $3 \mid m_{12}$ follows.
- $9 \mid (fm_{23} + 7(m_{12}m_{23} - fm_{13}) + 21fm_{24})$ for some $m_{24} \in \mathbb{Z}$ implies $3 \mid (m_{23} - m_{13})$.
- These conditions are sufficient.

Example application

The kernel of φ has the form

$$\{\begin{pmatrix} 1 & 0 & 0 & m_{14} & m_{15} \\ 0 & 1 & 0 & 3x & m_{25} \\ 0 & 0 & 1 & m_{34} & m_{35} \\ 0 & 0 & 0 & 1 & 7x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid x, m_{ij} \in \mathbb{Z}\}.$$