Max-Planck-Institut für Mathematik

Multiple harmonic q-series and finite & symmetrized MZV

MHS FMZV SMZV

j.w.
Y. Takeyama
K. Tasaka
[BTT] - MPIM 2017-43

A MHS is a sum of the form

$$\sum_{\mathrm{ord}(q) > m_1 > \dots > m_r > 0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \qquad Q_j(x) \in \mathbb{Q}[x], \quad k_j \in \mathbb{Z}_{\geq 1}.$$

qMZV

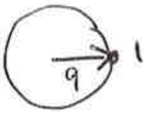
"classical case"

$\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

$|q| < 1$

Mod. forms $\subset$ MHS $\subset \mathcal{O}(\mathbb{H})$

$\downarrow$



MZV

Cusp forms $\rightsquigarrow$ Relations

$q = e^{2\pi i \tau} = e(\tau)$

"cyclotomic case": $\tau \in \mathbb{Q}$, $q^n = 1$ (n = ord(q))

3

$$\begin{tikzcd} & \mathrm{MHS} \subset \mathrm{Map}(\mathbb{Q}, \mathbb{C}) \arrow[dl, "{\text{“}q=1\text{”}}"', "\bmod 1-q"] \arrow[dr] & \\ \mathrm{FMZV} \arrow[rr, leftrightarrow, "\text{Kaneko-Zagier}"'] & & \mathrm{SMZV} \end{tikzcd}$$

"q→1"

2 1

① (S) MZV

$\mathbb{k} = (k_1, \dots, k_r)$, $k_1, \dots, k_r \geq 1$, $\mathrm{wt}(\mathbb{k}) = k_1 + \dots + k_r$

$\mathbb{k}$ adm. $\Leftrightarrow$ $k_1 \geq 2 \ \vee\ \mathbb{k} = \emptyset$ $(r=0)$

Def (MZV): For $\mathbb{k}$ adm. define

$$\zeta(\mathbb{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R} \qquad (\zeta(\emptyset) = 1)$$

- MZV satisfy many $\mathbb{Q}$-linear relations, ex: $\zeta(5) = 2\zeta(2,3) - 4\zeta(4,1)$.
- $\mathcal{Z} = \langle \zeta(\mathbb{k}) \mid \mathbb{k} \text{ adm.} \rangle_{\mathbb{Q}}$, $\mathcal{Z}_k = \langle \zeta(\mathbb{k}) \mid \mathbb{k} \text{ adm.}, \mathrm{wt}(\mathbb{k}) = k \rangle_{\mathbb{Q}}$.

Conjecture (Zagier)
$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k X^k = \frac{1}{1 - X^2 - X^3}$$

For any $\Bbbk$ let $\zeta(\Bbbk;T) \in \mathcal{Z}[T]$ denote the stuffle-regularized MZV.

Ex: $\zeta(1,2;T) = \zeta(2)T - 2\zeta(3)$.

Def (SMZV) For any $\Bbbk$ define

$$\zeta_S(\Bbbk) = \zeta_S(k_1,\dots,k_r) = \sum_{a=0}^{r} (-1)^{k_1+\dots+k_a} \zeta(k_a,\dots,k_1;T)\zeta(k_{a+1},\dots,k_r;T) \overset{\text{Prop.}}{\in} \mathcal{Z}$$

depth 1: $k \geq 1$ $\zeta_S(k) = \begin{cases} 2\zeta(k), & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$. $\zeta(3) = \frac{1}{3}\zeta_S(2,1)$

Thm (Yasuda 2014) $\mathcal{Z} = \langle \zeta_S(\Bbbk) \rangle_{\mathbb{Q}}$.

MZV relations $\Longrightarrow$ $\zeta_S(4,1) - \zeta_S(1,4) + \zeta_S(3,2) = 0$

2 FMZV

Def (FMZV) For any $\Bbbk$ define

$$\zeta_{\mathcal{A}}(\Bbbk) = \zeta_{\mathcal{A}}(k_1,\dots,k_r) = \left(\sum_{p > m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \bmod p \right)_{p \text{ prime}} \in \underbrace{\prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}}_{=:\mathcal{A}}$$

- $\mathcal{A}$ is a $\mathbb{Q}$-algebra
- depth 1: $k \geq 1$ $\zeta_{\mathcal{A}}(k) = 0$
- $\mathcal{Z}^{\mathcal{A}} = \langle \zeta_{\mathcal{A}}(\Bbbk) \rangle_{\mathbb{Q}}$, $\mathcal{Z}^{\mathcal{A}}_k = \langle \zeta_{\mathcal{A}}(\Bbbk) \mid \mathrm{wt}(\Bbbk) = k \rangle_{\mathbb{Q}}$
- Satisfy many linear relations, e.g.

$$\zeta_{\mathcal{A}}(4,1) - \zeta_{\mathcal{A}}(1,4) + \zeta_{\mathcal{A}}(3,2) = 0$$

Conjecture (Kaneko-Zagier) We have a $\mathbb{Q}$-algebra isomorphism

$$\mathcal{Z}^{\mathcal{A}} \longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}$$
$$\zeta_{\mathcal{A}}(\Bbbk) \longmapsto \zeta_S(\Bbbk) \bmod \pi^2\mathcal{Z},$$

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}^{\mathcal{A}}_k X^k = \frac{1-X^2}{1-X^2-X^3}.$$

③ MHS

$$\mathcal{I} = \left\{ (\mathbb{k};\mathbb{e}) \;\middle|\; \begin{array}{l} \mathbb{k}=(k_1,\dots,k_r) \quad k_j \geq 1 \\ \mathbb{e}=(e_1,\dots,e_r),\ r\geq 0,\ 0\leq e_j \leq k_j,\ j=1,\dots,r \end{array} \right\}$$

$$\cup$$

$$\mathcal{I}^0 = \{ (\mathbb{k};\mathbb{e})\in \mathcal{I} \mid 0\leq e_j \leq k_j - 1 \ \forall j=1,\dots,r \}.$$

$$\mathcal{I}_k = \{ (\mathbb{k};\mathbb{e}) \in \mathcal{I} \mid \mathrm{wt}(\mathbb{k}) = k \}, \quad \mathcal{I}_k^0 = \mathcal{I}^0 \cap \mathcal{I}_k .$$

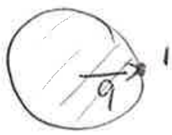
<u>Def</u> (MHS) For $\tau \in \mathbb{Q} \cup \mathbb{H}$ and $(\mathbb{k},\mathbb{e})\in\mathcal{I}$ $\left(\begin{smallmatrix} e_1\geq 1 \text{ if} \\ \tau\in\mathbb{H} \end{smallmatrix}\right)$ define

$q = e^{2\pi i \tau} = e(\tau)$

$$g(\mathbb{k};\mathbb{e};q) = g\left(\begin{smallmatrix} k_1,\dots,k_r \\ e_1,\dots,e_r \end{smallmatrix};q\right) = \sum_{\mathrm{ord}(q)>m_1>\dots>m_r>0} \frac{q^{m_1 e_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r e_r}}{(1-q^{m_r})^{k_r}} \in \mathbb{C}.$$

$$\mathrm{ord}(q) = \begin{cases} \infty &, \tau\in\mathbb{H} \\ \min\{n \mid q^n = 1\} &, \tau\in\mathbb{Q} \end{cases}$$

"classical case": $\tau\in\mathbb{H}$, $|q|<1$



- $k_1 \geq 2$: $\lim_{q\to 1} (1-q)^{\mathrm{wt}(\mathbb{k})} g(\mathbb{k};\mathbb{e};q) = \zeta(\mathbb{k})$

- $\sum_{(\mathbb{k},\mathbb{e})\in\mathcal{I}_k} \alpha_{\mathbb{k},\mathbb{e}} \, g(\mathbb{k};\mathbb{e};q) \in S_k \Longrightarrow \sum_{(\mathbb{k},\mathbb{e})\in\mathcal{I}_k} \alpha_{\mathbb{k},\mathbb{e}} \, \zeta(\mathbb{k}) = 0$

  $\alpha_{\mathbb{k},\mathbb{e}} \in \mathbb{Q}$

  S_k: Cusp forms of wt k

- Have - dimension conjectures
  - description of alg. structure
  - (conjectured) understanding of linear relations.

"cyclotomic case" : From now on $\tau \in \mathbb{Q}$, q root of unity
$(\mathbb{k}, \mathbb{e}) \in I$, $g(\mathbb{k}; \mathbb{e}) \in \mathrm{Map}(\mathbb{Q}, \mathbb{C})$.

$$\mathcal{G}^\circ = \langle g(\mathbb{k}; \mathbb{e}) \mid (\mathbb{k}, \mathbb{e}) \in I^\circ \rangle_{\mathbb{Q}}.$$

Thm I (BTT) For $(\mathbb{k}, \mathbb{e}) \in I^\circ$ we have

i) $$\xi(\mathbb{k}) := \lim_{n\to\infty} (1 - e(\tfrac{1}{n}))^{\mathrm{wt}(\mathbb{k})} g(\mathbb{k}; \mathbb{e}; e(\tfrac{1}{n})) = \sum_{a=0}^{r} (-1)^{k_1 + \dots + k_a} \zeta(k_a, \dots, k_1; \tfrac{\pi i}{2}) \zeta(k_{a+1}, \dots, k_r; -\tfrac{\pi i}{2}) \in \mathbb{Z} + \pi i \mathbb{Z}.$$

ii) $$\mathrm{Re}(\xi(\mathbb{k})) \equiv \zeta_S(\mathbb{k}) \mod \pi^2 \mathbb{Z}.$$

$$\left(\mathrm{Re}(\xi(\mathbb{k})) = \zeta_S(\mathbb{k}) \text{ if } \begin{array}{l} k_1, \dots, k_r \geq 2 \\ r = 1 \\ r = 2, \mathbb{k} \neq (1,1) \end{array} \right).$$

- Thm I gives a map $\varphi^S_K : \mathcal{G}^\circ_K \longrightarrow \mathcal{Z}_K$
$g(\mathbb{k}; \mathbb{e}) \longmapsto \mathrm{Re}(\xi(\mathbb{k}))$.

Thm II (BTT) For $(\mathbb{k}, \mathbb{e}) \in I^\circ$, p prime, $I_p = (1 - e(\tfrac{1}{p})) \subset \mathbb{Z}[e(\tfrac{1}{p})]$ we have

i) $$(1 - e(\tfrac{1}{p}))^{\mathrm{wt}(\mathbb{k})} g(\mathbb{k}; \mathbb{e}; e(\tfrac{1}{p})) \in \mathbb{Z}[e(\tfrac{1}{p})],$$

ii) $$\mathbb{Z}[e(\tfrac{1}{p})] / I_p \cong \mathbb{Z}/p\mathbb{Z},$$

iii) $$\zeta_A(\mathbb{k}) = \left((1 - e(\tfrac{1}{p}))^{\mathrm{wt}(\mathbb{k})} g(\mathbb{k}; \mathbb{e}; e(\tfrac{1}{p})) \mod I_p \right)_{p \text{ prime}}.$$

- Thm II gives a map $\varphi^A_K : \mathcal{G}^\circ_K \longrightarrow \mathcal{Z}^A_K$

- In $\mathcal{G}^\circ$ we can prove various relations, e.g.

$$g\binom{4,1}{0,0} - g\binom{1,4}{0,3} + \frac{1}{3} g\binom{3,2}{2,0} + \frac{2}{3} g\binom{3,2}{1,1} - \frac{1}{3} g\binom{3,2}{1,0} + \frac{4}{3} g\binom{3,2}{0,1} - g\binom{3,2}{0,0} = 0$$

$\swarrow \varphi^S_5$ $\searrow \varphi^A_5$

$$\zeta_S(4,1) - \zeta_S(1,4) + \zeta_S(3,2) = 0 \qquad \zeta_A(4,1) - \zeta_A(1,4) + \zeta_A(3,2) = 0.$$

Q: What are the kernels of $\varphi^S : \mathcal{G}^\circ \longrightarrow \mathcal{Z} \longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}$?

(Approach: Consider a larger space.) $\varphi^A : \mathcal{G}^\circ \longrightarrow \mathcal{Z}^A$

<u>Thm II'</u>: Thm II is also true for $(\mathbb{k},\mathbb{e}) \in I$.

$$\mathcal{G}_\kappa = \langle g(\mathbb{k};\mathbb{e}) \mid (\mathbb{k};\mathbb{e}) \in I_\kappa \rangle_{\mathbb{Q}} \xrightarrow{\varphi^A_\kappa} \mathcal{Z}^A_\kappa .$$

But: Thm I does not work for every $(\mathbb{k},\mathbb{e}) \in I$!

Ex: $\underset{(1,1)}{(\mathbb{k}};\underset{(1,0)}{\mathbb{e})} = (\underset{1,1}{\cancel{}};1.0) \in I \setminus I^\circ$

$$(1-e(\tfrac{1}{n}))^2 g\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}; e(\tfrac{1}{n})\right) = 2\mathcal{G}(2) + 2\pi i \left(\log\left(\tfrac{n}{2\pi}\right) - \gamma\right) + O\left(\tfrac{\log(n)}{n}\right)$$

as $n \to \infty$.

(i.e. $\varphi^S_\kappa(\mathbb{k};\mathbb{e})$ doesn't exist for all $(\mathbb{k},\mathbb{e}) \in I_\kappa$)

Now define

$$\widetilde{I} = I \setminus \{ ((\ldots,1,1,\ldots);(\ldots,1,0,\ldots)),\ (\ldots,1,1,\ldots;\ldots,0,1,\ldots) \}$$

$$\widetilde{\mathcal{G}} = \langle g(\mathbb{k};\mathbb{e}) \mid (\mathbb{k},\mathbb{e}) \in \widetilde{I} \rangle_{\mathbb{Q}} . \qquad \mathcal{G}^\circ \subset \widetilde{\mathcal{G}} \subset \mathcal{G}$$

<u>Observation I</u>: Can extend φ^S to $\widetilde{\mathcal{G}} \longrightarrow \mathcal{Z}$.

- Denote by $\mathcal{G}^\circ_\kappa$, $\widetilde{\mathcal{G}}_\kappa$ and $\mathcal{G}_\kappa$ the wt κ parts.
- The image of φ^A and φ^S is independent of $\mathbb{e}$, so we define:

$$\mathcal{E}_\kappa = \langle g(\mathbb{k};\mathbb{e}_1) - g(\mathbb{k};\mathbb{e}_2) \mid (\mathbb{k};\mathbb{e}_1), (\mathbb{k};\mathbb{e}_2) \in I_\kappa \rangle_{\mathbb{Q}}$$

$$\subset \ker \varphi^A_\kappa$$

similarly $\mathcal{E}^\circ_\kappa \subset \ker \varphi^A_\kappa$, $\subset \ker \varphi^S_\kappa$; $\widetilde{\mathcal{E}}_\kappa \subset \ker \varphi^A_\kappa$. $\underset{?}{\subset} \ker \varphi^S_\kappa$

$$\mathcal{Z}_k^{q,0} = \mathfrak{g}_k^0 / \mathcal{E}_k^0 \,, \quad \tilde{\mathcal{Z}}^q = \tilde{\mathfrak{g}}_k / \tilde{\mathcal{E}}_k \,, \quad \mathcal{Z}_k^q = \mathfrak{g}_k / \mathcal{E}_k \,.$$

Numerical data:

k	1	2	3	4	5	6	7	8
$\# I^0$	1	3	8	21	55	144	377	987
$\# \tilde{I}$	2	5	18	56	174	548	1720	5396
$\# I$	2	7	24	82	280	956	3264	11144
$\dim_{\mathbb{Q}} \mathfrak{g}_k^0 \overset{?}{=}$	1	2	4	6	10	15	23	
$\dim_{\mathbb{Q}} \tilde{\mathfrak{g}}_k \overset{?}{=}$	1	2	5	10	19	34	...	
$\dim_{\mathbb{Q}} \mathfrak{g}_k \overset{?}{=}$	1	3	7	13	25	45		
$\dim_{\mathbb{Q}} \mathcal{Z}_k^{q,0} \overset{?}{=}$	1	1	2	2	4	5	8	12
$\dim_{\mathbb{Q}} \tilde{\mathcal{Z}}_k^q$	0	1	1	1	2	2	3	4
$\dim_{\mathbb{Q}} \mathcal{Z}_k^q$	0	0	1	0	1	1	1	2

Observation II:
- $\dim_{\mathbb{Q}} \tilde{\mathcal{Z}}_k^q \overset{?}{=} \dim_{\mathbb{Q}} \mathcal{Z}_k$
- $\dim_{\mathbb{Q}} \mathcal{Z}_k^q \overset{?}{=} \dim_{\mathbb{Q}} \mathcal{Z}_k^A$
- $\ker \varphi^A \overset{?}{=} \mathcal{E}_k$.

Conjectural picture:

$$\begin{tikzcd}
 & \hat{\mathcal{G}} \arrow[d, "\pi"] & \\
 & \tilde{\mathcal{Z}}^q_K \arrow[dl, "\pi^q"'] \arrow[dr, "\varphi^S_K", "\sim"'] & \\
\mathcal{Z}^q_K \arrow[d, "\varphi^A"', "2"] & & \mathcal{Z}_K \arrow[d, "\pi"] \\
\mathcal{Z}^A_K & & (\mathcal{Z}/\pi^2\mathcal{Z})_K
\end{tikzcd}$$

$g\binom{2}{0} \mapsto \mathcal{Z}(2)$, $\mathcal{Z}(2) \mapsto 0 \mapsto 0$, $\mathcal{Z}(2) \mapsto 2\zeta(2) \mapsto 0$

π^q: $\tilde{\mathcal{G}}_K / \hat{\mathcal{E}}_K \hookrightarrow \mathcal{G}_K / \mathcal{E}_K$

Q: $\operatorname{Ker}(\varphi^A \circ \pi^q) \overset{?}{=} \operatorname{Ker}(\pi \circ \varphi^S)$.

Ex:

$$g\binom{2}{0} + g\binom{2}{1} = -g\begin{pmatrix}1&1\\0&0\end{pmatrix}$$

$$g\binom{2}{0} + g\binom{2}{1} = g\begin{pmatrix}1&1\\0&1\end{pmatrix} + g\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

$$\Rightarrow \quad g\binom{2}{0} \equiv g\begin{pmatrix}1&1\\0&0\end{pmatrix} \equiv 0 \mod \mathcal{E}_2$$

(but $\not\equiv 0 \mod \hat{\mathcal{E}}_2$.)