Invariant random subgroups II

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(A few are lines missing because of conversion to handout.)

Invariant random subgroups

Definition (Abért 2012)

A G-invariant probability measure ν on Sub_G is called an invariant random subgroup or IRS.

Theorem (Creutz-Peterson 2013)

If ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

Question

For example, can we classify the ergodic IRSs of Hall's universal locally finite group?

An application of the pointwise ergodic theorem

- Let $G = \bigcup G_n$ be locally finite and let $G \curvearrowright (Z, \mu)$ be ergodic.
- For each $z \in Z$ and $n \in \mathbb{N}$, let $\Omega_n(z) = \{ g \cdot z \mid g \in G_n \}$.

Theorem

With the above hypotheses, for μ-a.e. $z \in Z$, for all $g \in G$,

$$\mu(\mathrm{Fix}_Z(g)) = \lim_{n \to \infty} |\mathrm{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|.$$

Remark

Note that $|\mathrm{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|$ is the probability that an element of $(\Omega_n(z), \mu_n)$ is fixed by $g \in G_n$, where μ_n is the uniform probability measure on $\Omega_n(z)$

Characters of countable groups

Definition

If G is a countable group, then $\chi : G \to \mathbb{C}$ is a character if the following conditions are satisfied:

(i) $\chi(h\,g\,h^{-1}) = \chi(g)$ *for all* $g, h \in G$.

(ii) $\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ *for all* $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ *and* $g_1, \cdots, g_n \in G$.

(iii) $\chi(1_G) = 1$.

Example

If $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a probability space, then $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ is a character.

Characters of finite groups

Notation

- If G is a finite group, then $\widehat{G}$ denotes the set of (unitary equivalence classes) of irreducible representation of G.
- If $\pi \in \widehat{G}$, then $\chi^{\pi}(g) = \mathrm{trace}(\,\pi(g)\,)$.

Theorem

If G is a finite group and $\chi : G \to \mathbb{C}$ is a character, then χ is a convex combination of $\{\, \chi^{\pi}/\pi(1) \mid \pi \in \widehat{G} \,\}$.

Indecomposable characters

Definition

A character χ is indecomposable *if it is impossible to express*

$$\chi = r\chi_1 + (1 - r)\chi_2,$$

where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters.

Theorem (Thomas, October 2018)

The relationship between the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$ and its ergodic invariant random subgroups can be described precisely.

Inductive limits of finite alternating groups

Definition

- *G is an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$.*
- *Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.*

Definition

If ν is an IRS of the countable group G, then the associated character is defined by $\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \})$.

Theorem (Thomas 2018)

If G is an $L(\mathrm{Alt})$-group and $G \not\cong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are precisely the associated characters of its ergodic IRSs.

Trivial IRSs and characters of simple groups G

Definition

The trivial IRSs of G are δ_1 and δ_G.

Definition

The trivial characters of G are the regular character χ_{reg} and the constant character χ_{con}, where:

- $\chi_{reg}(g) = 0$ *for all* $1 \neq g \in G$*; and*
- $\chi_{con}(g) = 1$ *for all* $g \in G$.

Theorem (Kirillov 1965 & Peterson-Thom 2013)

If K is a countably infinite field and $n \geq 2$, then $PSL(n, K)$ is character rigid.

The Strong Simplicity Problem

Theorem (Peterson-Thom 2013 & Thomas-Tucker-Drob 2016)

If the countably infinite simple group G is character rigid, then G is strongly simple.

Question

Does there exist a strongly simple (locally finite) group which is not character rigid?

Remark

Strong simplicity and character rigidity coincide on the class of $L(\mathrm{Alt})$-groups.

Classification of the strongly simple $L(\mathrm{Alt})$-groups

Definition

Consider $G_i = \mathrm{Alt}(\Delta_i) \hookrightarrow G_{i+1} = \mathrm{Alt}(\Delta_{i+1})$ *and suppose that* $\Sigma \subseteq \Delta_{i+1}$ *is a* G_i*-orbit.*

- Σ *is **natural** if* $G_i \curvearrowright \Sigma$ *is isomorphic to* $\mathrm{Alt}(\Delta_i) \curvearrowright \Delta_i$*.*
- Σ *is **trivial** if* $|\Sigma| = 1$*.*
- *Otherwise,* Σ *is **exceptional**.*

Notation

- $n_i = |\Delta_i|$.
- s_{i+1} *is the number of natural* G_i*-orbits on* Δ_{i+1}*.*
- t_{i+1} *is the number of* $x \in \Delta_{i+1}$ *which lie in a trivial* G_i*-orbit.*
- e_{i+1} *is the number of* $x \in \Delta_{i+1}$ *which lie in an exceptional* G_i*-orbit.*

Classification of the strongly simple $L(\mathrm{Alt})$-groups

Definition (Zalesskii)

$G = \bigcup_{i \in \mathbb{N}} G_i$ *is a diagonal limit if* $e_{i+1} = 0$ *for all* $i \in \mathbb{N}$.

Theorem (Thomas-Tucker-Drob 2016)

If G is an $L(\mathrm{Alt})$*-group, then G has a nontrivial ergodic IRS if and only if G can be expressed as an almost diagonal limit of finite alternating groups.*

Definition

$G = \bigcup_{i \in \mathbb{N}} G_i$ *is an almost diagonal limit if*

- $s_{i+1} > 0$ *for all* $i \in \mathbb{N}$*, and*
- $\sum_{i=1}^{\infty} e_i / s_{0i} < \infty$,

where $s_{0i} = s_1 s_2 \cdots s_i$ *is the number of "obvious"* G_0*-orbits on* Δ_i.

Classifying the ergodic IRSs of diagonal limits

From now on, we suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a diagonal limit.

Definition

For each $i < j$, let $s_{ij} = s_{i+1} s_{i+2} \cdots s_j$ be the number of natural G_i-orbits on Δ_j.

The analysis initially splits into two cases:

- G has linear natural orbit growth.
- G has sublinear natural orbit growth.

Linear vs sublinear natural orbit growth

Proposition (Leinen-Puglisi 2003)

For each $i \in \mathbb{N}$, the limit $a_i = \lim_{j \to \infty} s_{ij}/n_j$ exists.

Proof.

If $i < j < k$, then $s_{ik} = s_{ij}s_{jk}$ and $s_{jk}n_j \leq n_k$. Hence we have that

$$\frac{s_{ik}}{n_k} = \frac{s_{ij}}{n_j} \cdot \frac{s_{jk}n_j}{n_k} \leq \frac{s_{ij}}{n_j}.$$

□

Definition (Leinen-Puglisi 2003)

G has linear natural orbit growth if $a_i > 0$ for some (equivalently every) $i \in \mathbb{N}$. Otherwise, G has sublinear natural orbit growth.

A natural candidate for a nontrivial ergodic IRS

Clearly we can suppose that

- $\Delta_0 = \{ \alpha^0_\ell \mid \ell < t_0 = n_0 \}$.
- $\Delta_{i+1} = \{ \sigma ^\frown k \mid \sigma \in \Delta_i, 0 \leq k < s_{i+1} \} \cup \{ \alpha^{i+1}_\ell \mid 0 \leq \ell < t_{i+1} \}$

and that the embedding $\varphi_i : \mathrm{Alt}(\Delta_i) \to \mathrm{Alt}(\Delta_{i+1})$ satisfies

- $\varphi_i(g)(\sigma ^\frown k) = g(\sigma) ^\frown k$
- $\varphi_i(g)(\alpha^{i+1}_\ell) = \alpha^{i+1}_\ell$

Let Δ consist of the infinite sequences of the form

$$(\alpha^i_\ell, k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$$

where $0 \leq k_j < s_j$. Then $G \curvearrowright \Delta$ via

$$g \cdot (\alpha^i_\ell, k_{i+1}, \cdots, k_j, k_{j+1} \cdots) = (g(\alpha^i_\ell, k_{i+1}, \cdots, k_j), k_{j+1} \cdots), \quad g \in G_j.$$

A natural candidate for a nontrivial ergodic IRS

For each $\sigma \in \Delta_i$, let $\Delta(\sigma) \subseteq \Delta$ be the set of sequences of the form

$$\sigma^\frown (\, k_{i+1}, k_{i+2}, k_{i+3}, \cdots).$$

Then the $\Delta(\sigma)$ form a clopen basis for a locally compact topology on Δ and $G \curvearrowright \Delta$ via homeomorphisms.

Question

When is there a G-invariant ergodic probability measure on Δ?

The Pointwise Ergodic Theorem

Theorem (Vershik 1974 & Lindenstrauss 1999)

Suppose that $G = \bigcup G_i$ *is locally finite and that* $G \curvearrowright (Z, \mu)$ *is ergodic. If* $B \subseteq Z$ *is* μ*-measurable, then for* μ*-a.e* $z \in Z$*,*

$$\mu(B) = \lim_{i \to \infty} \frac{1}{|G_i|} |\{ g \in G_i \mid g \cdot z \in B \}|.$$

Linear vs sublinear natural orbit growth

Proposition

If μ is a G-invariant ergodic probability measure on Δ and $\sigma \in \Delta_i$, then

$$\mu(\Delta(\sigma)) = \lim_{j \to \infty} s_{ij}/n_j = a_i.$$

Corollary

If G has sublinear natural orbit growth, then no such μ exists.

Proof.

Supposing that μ exists, we have that

$$1 = \mu(\Delta) = \sum_{i \in \mathbb{N}} \sum_{0 \leq \ell < t_i} \mu(\Delta(\alpha_\ell^i)) = 0.$$

□

The proof of the proposition

- Let $\sigma \in \Delta_i$ and choose $z \in \Delta$ such that
$$\mu(\Delta(\sigma)) = \lim_{j \to \infty} \frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot z \in \Delta(\sigma) \}|.$$
- Let $z = (\alpha^r_\ell, k_{r+1}, k_{r+2}, k_{r+3}, \cdots)$ and for each $j > r$, let
$$z_j = (\alpha^r_\ell, k_{r+1}, k_{r+2}, k_{r+3}, \cdots , k_j) \in \Delta_j.$$
- For each $j > \max\{ i, r \}$, let $S_j \subseteq \Delta_j$ be the set of sequences of the form $\sigma \,\widehat{}\, (d_{i+1}, d_{i+2}, \cdots , d_j)$.
- Then $|S_j| = s_{ij}$ and we have that
$$\{ g \in G_j \mid g \cdot z \in \Delta(\sigma) \} = \{ g \in G_j \mid g \cdot z_j \in S_j \}.$$
- It now follows that
$$\mu(\Delta(\sigma)) = \lim_{j \to \infty} \frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot z_j \in S_j \}| = \lim_{j \to \infty} |S_j|/|\Delta_j| = a_i.$$

The ergodic IRS's for linear natural orbit growth

Theorem

If G has linear natural orbit growth, then there exists a unique G-invariant ergodic probability measure μ on Δ.

Non-obvious Corollary

If G has linear natural orbit growth, then the diagonal action $G \curvearrowright (\,\Delta^r, \mu^{\otimes r}\,)$ is ergodic for all $r \geq 1$.

Theorem (Thomas-Tucker-Drob 2016)

If G has linear natural orbit growth and $\nu \neq \delta_1, \delta_G$ is an ergodic IRS, then there exists $r \geq 1$ such that ν is the stabilizer distribution of $G \curvearrowright (\,\Delta^r, \mu^{\otimes r}\,)$.

The basic strategy for groups of linear orbit growth

- Suppose that $G = \bigcup G_i$ has linear natural orbit growth.
- Let ν be a nontrivial ergodic IRS of G.
- Then ν is the stabilizer distribution of an ergodic $G \curvearrowright (Z, \mu)$.

Observation

If ν_r is the stabilizer distribution of $G \curvearrowright (\Delta^r, \mu^{\otimes r})$, then for ν_r-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $i \in \mathbb{N}$, there exists $\Sigma_i \subset \Delta_i$ with $|\Delta_i \smallsetminus \Sigma_i| = r$ such that $H_i = H \cap \mathrm{Alt}(\Delta_i) = \mathrm{Alt}(\Sigma_i)$.

- Choose a μ-random point $z \in Z$ and let $H = \{ h \in G \mid h \cdot z = z \}$ be the corresponding ν-random subgroup.
- Then $H \neq G$ and $\mu(\mathrm{Fix}_Z(h)) > 0$ for all $h \in H$.

Another application of the pointwise ergodic theorem

- For each $i \in \mathbb{N}$, let $\Omega_i = \{ g \cdot z \mid g \in G_i \}$.
- Note that $G_i \curvearrowright \Omega_i$ is isomorphic to $G_i \curvearrowright G_i / H_i$, where $H_i = \{ h \in G_i \mid h \cdot z = z \} = H \cap G_i$.

Theorem

Since $z \in Z$ is μ-random, for all $g \in G$,

$$\begin{aligned} \mu(\operatorname{Fix}_Z(g)) &= \lim_{i \to \infty} |\operatorname{Fix}_{\Omega_i}(g)| / |\Omega_i| \\ &= \lim_{i \to \infty} \frac{|g^{G_i} \cap H_i|}{|g^{G_i}|}. \end{aligned}$$

$H_i \curvearrowright \Delta_i$ is primitive for only finitely many $i \in \mathbb{N}$

- Let $h \in H$ be an element of prime order p.
- Regarded as an element of $\mathrm{Alt}(\Delta_i)$, let h be a product of c_i p-cycles.
- Since G has linear natural orbit growth, there exists a constant $b > 0$ such that $c_i \geq b\, n_i$.
- Hence there exist constants $r, s > 0$ such that

$$|h^{\mathrm{Alt}(\Delta_i)}| > r\, s^{n_i}\, n_i^{n_i(p-1)b}$$

$H_i \curvearrowright \Delta_i$ is primitive for only finitely many $i \in \mathbb{N}$

- Suppose that $H_i \curvearrowright \Delta_i$ is primitive for infinitely many $i \in \mathbb{N}$.

Theorem (Praeger-Saxl 1979)

If $H_i < \mathrm{Alt}(n_i)$ is a proper primitive subgroup, then $|H_i| < 4^{n_i}$.

- But this means that

$$\begin{aligned}\mu(\,\mathrm{Fix}_Z(h)\,) &= \lim_{i\to\infty} \frac{|\,h^{\mathrm{Alt}(\Delta_i)} \cap H_i\,|}{|\,h^{\mathrm{Alt}(\Delta_i)}|} \\ &\leq \lim_{i\to\infty} \frac{|\,H_i\,|}{|\,h^{\mathrm{Alt}(\Delta_i)}|} = 0,\end{aligned}$$

which is a contradiction!

How about groups of sublinear natural orbit growth?

Basic Idea

Construct an IRS ν *which concentrates on subgroups*

$$H = \bigcup \mathrm{Alt}(\Sigma_i), \qquad \Sigma_i \subset \Delta_i,$$

such that $|\Delta_i \smallsetminus \Sigma_i| \to \infty$ *at a "suitable rate".*

- Let Σ consist of the sequences $(\Sigma_i)_{i \in \mathbb{N}}$ such that:
  - $\Sigma_i \subseteq \Delta_i$
  - $\mathrm{Alt}(\Sigma_{i+1}) \cap G_i = \mathrm{Alt}(\Sigma_i)$.
- For each $X \subseteq \Delta_i$, let $\Sigma(X) \subseteq \Sigma$ be the sequences such that $\Sigma_i = X$.
- Then the $\Sigma(X)$ form a basis for a locally compact topology on Σ and $G \curvearrowright \Sigma$ via homeomorphisms.

Define a G-invariant probability measure μ on Σ

- For each $X \subseteq \Delta_i$, we would like to define

$$\mu(\,\Sigma(X)\,) = \left(\,1/e^{\beta_i}\,\right)^{|X|}\left(\,1 - 1/e^{\beta_i}\,\right)^{n_i - |X|},$$

  for some "suitably chosen" $\beta_i \in \mathbb{R}^+$.

- Then we must have $1/e^{\beta_i} = (1/e^{\beta_{i+1}})^{s_{i+1}}$.

- Thus $\beta_{i+1} = \beta_i / s_{i+1} = \beta_0 / s_1 s_2 \cdots s_{i+1} = \beta_0 / s_{0i+1}$.

Theorem

$\mu = \mu_{\beta_0}$ is a G-invariant probability measure μ on Σ.

But when is μ_{β_0} ergodic?

Proposition

If G has linear natural orbit growth, then μ_{β_0} is not ergodic.

Proof.

If $\sigma = (\Delta_i)_{i \in \mathbb{N}}$, then $\{\sigma\}$ is G-invariant. Furthermore,

$$\begin{aligned}
\mu_{\beta_0}(\{\sigma\}) &= \lim_{i \to \infty} \mu_{\beta_0}(\Sigma(\Delta_i)) \\
&= \lim_{i \to \infty} \frac{1}{e^{\beta_i n_i}} \\
&= \lim_{i \to \infty} \frac{1}{e^{\beta_0 n_i / s_{0i}}} \\
&= \frac{1}{e^{\beta_0 / a_0}}
\end{aligned}$$

□

The ergodic IRS's for sublinear natural orbit growth

Theorem

If G has sublinear natural orbit growth, then μ_{β_0} is ergodic.

Theorem (Thomas-Tucker-Drob 2016)

If $G \neq \mathrm{Alt}(\mathbb{N})$ has sublinear natural orbit growth and $\nu \neq \delta_1, \delta_G$ is an ergodic IRS, then there exists $\beta_0 \in \mathbb{R}^+$ such that ν is the stabilizer distribution of $G \curvearrowright (\Sigma, \mu_{\beta_0})$.

Theorem (Vershik 2012)

$\mathrm{Alt}(\mathbb{N})$ has a much richer collection of ergodic IRS's.

The End