Invariant random subgroups I

Simon Thomas

Rutgers University

5th November 2018
Guy Fawkes Day

(A few are lines missing because of conversion to handout.)

The space of subgroups

- Let G be a countable group and let

$$\text{Sub}_G \subset \mathcal{P}(G) = \{\,0,1\,\}^G = 2^G$$

be the set of subgroups $H \leqslant G$.

Observation

Sub_G is a closed subset of 2^G.

Proof.

If $S \in 2^G$ isn't a subgroup, then either

- $S \in \{\, T \in 2^G \mid 1 \notin T\}$,

or there exist $a, b \in G$ such that

- $S \in \{\, T \in 2^G \mid a, b \in T \text{ and } ab^{-1} \notin T\}$.

□

Invariant random subgroups

- Note that $G \curvearrowright \mathrm{Sub}_G$ via conjugation: $H \overset{g}{\mapsto} g\,H\,g^{-1}$.

Definition (Abért)

A G-invariant probability measure ν on Sub_G is called an invariant random subgroup or IRS.

A Boring Example

If $N \trianglelefteq G$, then the Dirac measure δ_N is an IRS of G.

Stabilizer distributions

Observation

- Suppose that $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a probability space.
- Let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant map defined by $z \mapsto G_z = \{ g \in G \mid g \cdot z = z \}$.
- Then the stabilizer distribution $\nu = f_* \mu$ is an IRS of G.
- If $B \subseteq \mathrm{Sub}_G$, then $\nu(B) = \mu(\{ z \in Z \mid G_z \in B \})$.

Theorem (Abért-Glasner-Virag 2012)

If ν is an IRS of G, then ν is the stabilizer distribution of a measure-preserving action $G \curvearrowright (Z, \mu)$.

Ergodicity

Definition

A measure-preserving action $G \curvearrowright (Z, \mu)$ is ergodic if $\mu(A) = 0$, 1 for every G-invariant μ-measurable subset $A \subseteq Z$.

Theorem

If $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a probability space, then the following statements are equivalent.

- *$G \curvearrowright (Z, \mu)$ is ergodic.*
- *If Y is a standard Borel space and $f : Z \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq Z$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

Remark

If ν is an ergodic IRS of G, then for every group-theoretic property Φ,

$$\nu(\{ H \in \mathrm{Sub}_G \mid H \text{ satisfies } \Phi \}) \in \{0, 1\}.$$

Observation

If $G \curvearrowright (Z, \mu)$ is ergodic, then the corresponding stabilizer distribution ν is an ergodic IRS of G.

Theorem (Creutz-Peterson 2013)

If ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

The Classification Problem

Problem

Given a countable group G, explicitly classify the ergodic IRSs of G (or show that no such classification is possible).

Theorem (Kirillov 1965 & Peterson-Thom 2013)

If K is a countably infinite field and $n \geq 2$, then the only ergodic IRS of $G = PSL(n, K)$ are δ_1 and δ_G.

Definition

A countable group G is strongly simple if the only ergodic IRS of G are δ_1 and δ_G.

What about $G = SL(3, \mathbb{Z})$?

Example

- For each $H \in \mathrm{Sub}_G$, let $\mathcal{C}(H) = \{\, gHg^{-1} \mid g \in G \,\}$.
- If $[\, G : H \,] < \infty$, then $|\mathcal{C}(H)| < \infty$ and we can define an ergodic IRS by

$$\nu_H(gHg^{-1}) = 1/|\mathcal{C}(H)|.$$

Theorem (Stuck-Zimmer 1994 & Abért-Glasner-Virag 2012)

The ergodic IRSs of $G = SL(3, \mathbb{Z})$ are precisely

$$\{\, \delta_1 \,\} \cup \{\, \nu_H \mid [\, G : H \,] < \infty \,\}.$$

What about $G = SL(3, \mathbb{Z})$?

- Suppose that $\nu \neq \delta_1$ is an ergodic IRS of $G = SL(3, \mathbb{Z})$.
- By Creutz-Peterson, ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.
- Clearly $G \curvearrowright (Z, \mu)$ is not essentially free.
- By Stuck-Zimmer, this implies that there exists an orbit $G \cdot z$ such that $\mu(G \cdot z) = 1$.
- Then $|G \cdot z| < \infty$ and $\nu = \nu_{G_z}$.

Some unclassifiable examples ...

Definition

If G is a countable group, then $M(G)$ denotes the simplex of invariant random subgroups of G.

Theorem (Bowen 2012)

If G is a nonabelian free group, then $M(G)$ has a canonical Poulsen subsimplex.

Theorem (Bowen-Grigorchuk-Kravchenko 2012)

If $G = (C_p)^n \operatorname{wr} \mathbb{Z}$ is a lamplighter group, then $M(G)$ has a canonical Poulsen subsimplex.

A more interesting example ...

Definition

The finitary symmetric group $\mathrm{Fin}(\mathbb{N})$ *is the subgroup of permutations* $\pi \in \mathrm{Sym}(\mathbb{N})$ *such that* $\{ n \in \mathbb{N} \mid \pi(n) \neq n \}$ *is finite.*

Theorem (Vershik 2011)

The uncountably many ergodic IRSs of $G = \mathrm{Fin}(\mathbb{N})$ *can be explicitly classified.*

Remark

Throughout we will identify partitions of $\mathbb{N}$ with the corresponding equivalence relations.

Ergodic Random Invariant Partitions

- Let $\alpha = (\alpha_i) \in [0,1]^{\mathbb{N}}$ be a sequence such that:
  - $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \geq 0$
  - $\sum_{i=0}^{\infty} \alpha_i = 1$.
- Define a probability measure p_α on $\mathbb{N}$ by $p_\alpha(\{i\}) = \alpha_i$.
- Let μ_α be the corresponding product probability measure on $\mathbb{N}^{\mathbb{N}}$.
- Then $\mathrm{Fin}(\mathbb{N}) \curvearrowright (\mathbb{N}^{\mathbb{N}}, \mu_\alpha)$ acts ergodically via the shift action $(\pi \cdot \xi)(n) = \xi(\pi^{-1}(n))$.
- Let $B_i^\xi = \{ n \in \mathbb{N} \mid \xi(n) = i \}$.
- Let $\xi \stackrel{\varphi}{\mapsto} \mathcal{P}^\xi = \bigsqcup_{n \in B_0^\xi} \{ n \} \sqcup \bigsqcup_{i > 0} B_i^\xi$.
- Then $m_\alpha = \varphi_* \mu_\alpha$ is an ergodic random invariant partition of $\mathbb{N}$.

Kingman's Theorem

Theorem (Kingman 1978)

If m is an ergodic random invariant partition of $\mathbb{N}$, then there exists α as above such that $m = \varphi_* \mu_\alpha$.

Observation (The law of large numbers)

For μ_α-a.e. $\xi \in \mathbb{N}^{\mathbb{N}}$, the following are equivalent for all $i \in \mathbb{N}^+$.

(a) $\alpha_i > 0$.

(b) $B_i^\xi \neq \emptyset$.

(c) B_i^ξ is infinite.

(d) $\lim_{n\to\infty} |\{\, \ell \in n \mid \xi(\ell) = i \,\}|/n = \alpha_i > 0$.

In this case, we say that ξ is μ_α-random.

The ergodic IRSs of $\mathrm{Fin}(\mathbb{N})$

Example

Suppose that $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$. Let ξ be μ_α-generic. Then there are the following obvious possibilities for a corresponding random subgroup.

(i) $H_\xi = \mathrm{Fin}(B_1^\xi) \times \mathrm{Fin}(B_2^\xi)$.

(ii) $H_\xi = \mathrm{Alt}(B_1^\xi) \times \mathrm{Alt}(B_2^\xi)$.

(iii) $H_\xi = \mathrm{Fin}(B_1^\xi) \times \mathrm{Alt}(B_2^\xi)$.

(iv) $H_\xi = \mathrm{Alt}(B_1^\xi) \times \mathrm{Fin}(B_2^\xi)$.

(v) $H_\xi = \{ (\pi, \theta) \in \mathrm{Fin}(B_1^\xi) \times \mathrm{Fin}(B_2^\xi) \mid \mathrm{sgn}(\pi) = \mathrm{sgn}(\theta) \}$.

The ergodic IRSs of $\mathrm{Fin}(\mathbb{N})$

- Let $I = \{\, i \in \mathbb{N}^+ \mid \alpha_i > 0 \,\}$ and let $S_\alpha = \bigoplus_{i \in I} C_i$, where each $C_i = \{\, \pm 1 \,\}$ is cyclic of order 2.
- Fix some subgroup $A \leqslant S_\alpha$.
- Let ξ be μ_α-random and let s_ξ be the homomorphism

$$s_\xi : \bigoplus_{i \in I} \mathrm{Fin}(B_i^\xi) \to \bigoplus_{i \in I} C_i$$
$$(\, \pi_i \,) \mapsto (\, \mathrm{sgn}(\pi_i) \,).$$

- Let $\xi \overset{f^A}{\mapsto} H_\xi = s_\xi^{-1}(A)$.
- Then $\nu_\alpha^A = (f^A)_* \mu_\alpha$ is an ergodic IRS of $\mathrm{Fin}(\mathbb{N})$.

The ergodic IRSs of $\mathsf{Fin}(\mathbb{N})$

Theorem (Vershik 2011 with corrections by Thomas 2013)

If ν is an ergodic IRS of $\mathsf{Fin}(\mathbb{N})$, then there exists α, A as above such that $\nu = \nu_\alpha^A$.

- The proof makes use of:

Theorem (Wielandt 1959)

If H $\leqslant \mathsf{Fin}(\mathbb{N})$ is a primitive subgroup, then H $= \mathsf{Alt}(\mathbb{N}), \mathsf{Fin}(\mathbb{N})$.

The proof begins ...

- Suppose that ν is an ergodic IRS of $\mathsf{Fin}(\mathbb{N})$.
- For each $H \in \mathrm{Sub}_{\mathsf{Fin}(\mathbb{N})}$, let $p(H)$ be the partition of $\mathbb{N}$ into H-orbits.
- Then $m = p_*\nu$ is an ergodic random invariant partition of $\mathbb{N}$.
- Hence there exists $\alpha \in [\,0, 1\,]^{\mathbb{N}}$ such that $m = \varphi_* \mu_\alpha$.
- Let $I = \{\, i \in \mathbb{N}^+ \mid \alpha_i > 0 \,\}$.
- Then for ν-a.e. $H \in \mathrm{Sub}_{\mathsf{Fin}(\mathbb{N})}$, there exists a μ_α-generic $\xi \in \mathbb{N}^{\mathbb{N}}$ such that $p(H)$ is
$$\mathbb{N} = \bigsqcup_{i \in I} B_i^{\xi} \sqcup \bigsqcup_{n \in B_0^{\xi}} \{\, n \,\}.$$

An application of Wielandt's Theorem

Lemma

For ν-a.e. $H \in Sub_{\mathrm{Fin}(\mathbb{N})}$, for all $i \in I$, the subgroup H induces at least $\mathrm{Alt}(B_i^\xi)$ on B_i^ξ.

- If not, then there exists a fixed $i \in I$ such that for ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, the subgroup H preserves a nontrivial equivalence relation E_B on $B = B_i^\xi$.
- Clearly each E_B-class is finite.

Claim

We can choose E_B in a $\mathrm{Fin}(\mathbb{N})$-equivariant Borel manner.

Assuming the Claim ...

- The Borel map $H \overset{e}{\mapsto} E_H = E_B \sqcup \mathrm{Id}(\mathbb{N} \smallsetminus B)$ is $\mathrm{Fin}(\mathbb{N})$-equivariant.
- Thus $m' = e_* \nu$ is an ergodic random invariant partition which concentrates on the equivalence relations E such that
  - every E-class is finite;
  - there exists an E-class of some fixed size $k > 1$.
- For each $S \in [\,\mathbb{N}\,]^k$, let C_S be the event that S is an E-class.
- Then there exists a fixed $r > 0$ such that $m'(C_S) = r$ for all $S \in [\,\mathbb{N}\,]^k$.
- Since the events $\{\, C_S \mid 0 \in S \in [\,\mathbb{N}\,]^k \,\}$ are mutually exclusive, this is a contradiction.

Lemma

If $H \leqslant \mathrm{Fin}(\mathbb{N})$ acts transitively but imprimitively on $\mathbb{N}$, then there exist only finitely many minimal nontrivial H-invariant equivalence relations.

- Suppose that $\{ E_n \mid n \in \mathbb{N} \}$ are distinct minimal H-invariant nontrivial equivalence relations.
- Let C_n be the E_n-class such that $0 \in C_n$.
- If $n \neq m$, then $C_n \cap C_m = \{ 0 \}$.
- Choose $a_n \in C_n \smallsetminus \{ 0 \}$.
- Then there exists $\pi \in H$ such that $\pi(0) = a_0$.
- If $n > 0$, then $a_0 \in \pi(C_n) \neq C_n$ and so $\pi(a_n) \neq a_n$.
- But this means that $\pi \notin \mathrm{Fin}(\mathbb{N})$.

Proof of the Claim

Theorem (Neumann 1975 & Segal 1974)

If $H \leqslant \mathrm{Fin}(\mathbb{N})$ acts transitively but imprimitively on $\mathbb{N}$, then either:

(i) *there exists a unique maximal nontrivial H-invariant equivalence relation E_{max}; or*

(ii) *there exists a sequence of nontrivial H-invariant equivalence relations $R_0 \subset R_1 \subset \cdots \subset R_\ell \subset \cdots$ such that $\mathbb{N}^2 = \bigcup R_\ell$.*

- If (i) holds, we can choose E_{max}.
- Suppose that (ii) holds and let $E_1, \cdots, E_n$ be the minimal nontrivial H-invariant equivalence relations.
- Then there exists a unique nontrivial H-invariant equivalence relation R which is minimal such that $E_1 \cup \cdots \cup E_n \subseteq R$.

The analysis continues ...

Lemma

For ν-a.e. $H \in Sub_{\mathrm{Fin}(\mathbb{N})}$, for all $i \in I$, the subgroup H induces at least $\mathrm{Alt}(B_i^\xi)$ on B_i^ξ.

Lemma

For ν-a.e. $H \in Sub_{\mathrm{Fin}(\mathbb{N})}$, we have that $\bigoplus_{i \in I} \mathrm{Alt}(B_i^\xi) \leqslant H$.

By a purely algebraic argument ...

Proposition

There exists a partition $\{ F_\alpha \mid \alpha \in \Omega \}$ *of* I *into finite pieces such that*

$$H \cap \bigoplus_{i \in I} \mathrm{Alt}(B_i^\xi) = \bigoplus_{\alpha \in \Omega} \mathrm{Diag}(\bigoplus_{j \in F_\alpha} \mathrm{Alt}(B_j^\xi)),$$

where the diagonal subgroups are determined by unique bijections $T_{jk} : B_j^\xi \to B_k^\xi$ *for* $j, k \in F_\alpha$.

- Let E be the ergodic invariant random partition defined by

$$n \, E \, m \iff (\exists j, k) \; T_{jk}(n) = m.$$

- Since E has finite classes, it follows that each $|F_\alpha| = 1$.

Almost there ...

Remark

An easy ergodicity argument shows that there exists a fixed subgroup $A \leqslant \bigoplus_{i \in I} C_i$ such that ν concentrates on the same collection $X_\alpha^A \subseteq \mathrm{Sub}_G$ of subgroups as ν_α^A.

Question

But why is $\nu = \nu_\alpha^A$*?*

Theorem

The action $\mathrm{Fin}(\mathbb{N}) \curvearrowright X_\alpha^A$ *is uniquely ergodic.*

The Pointwise Ergodic Theorem

Theorem (Vershik 1974 & Lindenstrauss 1999)

Suppose that $G = \bigcup_{n \in \mathbb{N}} G_n$ *is the union of an increasing chain of finite groups and that* $G \curvearrowright (Z, \mu)$ *is ergodic. Then for* μ*-a.e.* $z \in Z$*, for every* μ*-measurable subset* $Y \subseteq Z$*,*

$$\mu(Y) = \lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z \in Y \}|.$$

Remark

Thus, to prove that $G \curvearrowright (Z, \mu)$ is uniquely ergodic, it is enough to show that if $z, z' \in Z$ and $U \subseteq Z$ is a basic open subset, then

$$\lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z \in U \}| = \lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z' \in U \}|.$$

The End