Invariant random subgroups I

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Guy Fawkes Day

(A few are lines missing because of conversion to handout.)
The space of subgroups

- Let $G$ be a countable group and let
  \[ \text{Sub}_G \subset \mathcal{P}(G) = \{0, 1\}^G = 2^G \]
  be the set of subgroups $H \leq G$.

Observation

Sub$_G$ is a closed subset of $2^G$.

Proof.

If $S \in 2^G$ isn’t a subgroup, then either

- $S \in \{T \in 2^G \mid 1 \notin T\}$,

or there exist $a, b \in G$ such that

- $S \in \{T \in 2^G \mid a, b \in T \text{ and } ab^{-1} \notin T\}$.
Note that $G \sim \text{Sub}_G$ via conjugation: $H \leftrightarrow gHg^{-1}$.

**Definition (Abért)**

A $G$-invariant probability measure $\nu$ on $\text{Sub}_G$ is called an **invariant random subgroup** or IRS.

**A Boring Example**

If $N \trianglelefteq G$, then the Dirac measure $\delta_N$ is an IRS of $G$. 
Stabilizer distributions

Observation

- Suppose that \( G \curvearrowright (Z, \mu) \) is a measure-preserving action on a probability space.
- Let \( f : Z \to \text{Sub}_G \) be the \( G \)-equivariant map defined by \( z \mapsto G_z = \{ g \in G \mid g \cdot z = z \} \).
- Then the stabilizer distribution \( \nu = f_* \mu \) is an IRS of \( G \).
- If \( B \subseteq \text{Sub}_G \), then \( \nu(B) = \mu(\{ z \in Z \mid G_z \in B \}) \).

Theorem (Abért-Glasner-Virag 2012)

*If \( \nu \) is an IRS of \( G \), then \( \nu \) is the stabilizer distribution of a measure-preserving action \( G \curvearrowright (Z, \mu) \).*
Ergodicity

Definition
A measure-preserving action $G \curvearrowright (Z, \mu)$ is ergodic if $\mu(A) = 0, 1$ for every $G$-invariant $\mu$-measurable subset $A \subseteq Z$.

Theorem
If $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a probability space, then the following statements are equivalent.

- $G \curvearrowright (Z, \mu)$ is ergodic.
- If $Y$ is a standard Borel space and $f : Z \to Y$ is a $G$-invariant Borel function, then there exists a $G$-invariant Borel subset $M \subseteq Z$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function.
Ergodicity

Remark
If \( \nu \) is an ergodic IRS of \( G \), then for every group-theoretic property \( \Phi \),

\[
\nu( \{ H \in \text{Sub}_G \mid H \text{ satisfies } \Phi \} ) \in \{ 0, 1 \}.
\]

Observation
If \( G \curvearrowright (Z, \mu) \) is ergodic, then the corresponding stabilizer distribution \( \nu \) is an ergodic IRS of \( G \).

Theorem (Creutz-Peterson 2013)
If \( \nu \) is an ergodic IRS of \( G \), then \( \nu \) is the stabilizer distribution of an ergodic action \( G \curvearrowright (Z, \mu) \).
The Classification Problem

**Problem**

Given a countable group $G$, explicitly classify the ergodic IRSs of $G$ (or show that no such classification is possible).

**Theorem (Kirillov 1965 & Peterson-Thom 2013)**

*If $K$ is a countably infinite field and $n \geq 2$, then the only ergodic IRS of $G = \text{PSL}(n, K)$ are $\delta_1$ and $\delta_G$.***

**Definition**

A countable group $G$ is strongly simple if the only ergodic IRS of $G$ are $\delta_1$ and $\delta_G$. 
What about $G = SL(3, \mathbb{Z})$?

Example

- For each $H \in \text{Sub}_G$, let $\mathcal{C}(H) = \{ gHg^{-1} \mid g \in G \}$.
- If $[G : H] < \infty$, then $|\mathcal{C}(H)| < \infty$ and we can define an ergodic IRS by

$$\nu_H(gHg^{-1}) = 1/|\mathcal{C}(H)|.$$  

Theorem (Stuck-Zimmer 1994 & Abért-Glasner-Virag 2012)

The ergodic IRSs of $G = SL(3, \mathbb{Z})$ are precisely

$$\{ \delta_1 \} \cup \{ \nu_H \mid [G : H] < \infty \}.$$
What about \( G = SL(3, \mathbb{Z}) \)?

- Suppose that \( \nu \neq \delta_1 \) is an ergodic IRS of \( G = SL(3, \mathbb{Z}) \).
- By Creutz-Peterson, \( \nu \) is the stabilizer distribution of an ergodic action \( G \curvearrowright (Z, \mu) \).
- Clearly \( G \curvearrowright (Z, \mu) \) is not essentially free.
- By Stuck-Zimmer, this implies that there exists an orbit \( G \cdot z \) such that \( \mu(G \cdot z) = 1 \).
- Then \( |G \cdot z| < \infty \) and \( \nu = \nu_{Gz} \).
Some unclassifiable examples ...

**Definition**
If $G$ is a countable group, then $M(G)$ denotes the simplex of invariant random subgroups of $G$.

**Theorem (Bowen 2012)**
If $G$ is a nonabelian free group, then $M(G)$ has a canonical Poulsen subsimplex.

**Theorem (Bowen-Grigorchuk-Kravchenko 2012)**
If $G = (C_p)^n \wr \mathbb{Z}$ is a lamplighter group, then $M(G)$ has a canonical Poulsen subsimplex.
A more interesting example ...

Definition

The finitary symmetric group $\text{Fin}(\mathbb{N})$ is the subgroup of permutations $\pi \in \text{Sym}(\mathbb{N})$ such that $\{ n \in \mathbb{N} \mid \pi(n) \neq n \}$ is finite.

Theorem (Vershik 2011)

The uncountably many ergodic IRSs of $G = \text{Fin}(\mathbb{N})$ can be explicitly classified.

Remark

Throughout we will identify partitions of $\mathbb{N}$ with the corresponding equivalence relations.
Let \( \alpha = ( \alpha_i ) \in [0, 1]^\mathbb{N} \) be a sequence such that:

- \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \geq 0 \)
- \( \sum_{i=0}^{\infty} \alpha_i = 1 \).

Define a probability measure \( p_{\alpha} \) on \( \mathbb{N} \) by \( p_{\alpha}(\{i\}) = \alpha_i \).

Let \( \mu_{\alpha} \) be the corresponding product probability measure on \( \mathbb{N}^\mathbb{N} \).

Then \( \text{Fin}(\mathbb{N}) \bowtie (\mathbb{N}^\mathbb{N}, \mu_{\alpha}) \) acts ergodically via the shift action \( (\pi \cdot \xi)(n) = \xi(\pi^{-1}(n)) \).

Let \( B_i^\xi = \{ n \in \mathbb{N} \mid \xi(n) = i \} \).

Let \( \xi \mapsto \mathcal{P}_{\xi} = \bigsqcup_{n \in B_0^\xi} \{ n \} \sqcup \bigsqcup_{i > 0} B_i^\xi \).

Then \( m_{\alpha} = \varphi \cdot \mu_{\alpha} \) is an ergodic random invariant partition of \( \mathbb{N} \).
Kingman’s Theorem

Theorem (Kingman 1978)

If \( m \) is an ergodic random invariant partition of \( \mathbb{N} \), then there exists \( \alpha \) as above such that \( m = \varphi_\ast \mu_\alpha \).

Observation (The law of large numbers)

For \( \mu_\alpha \)-a.e. \( \xi \in \mathbb{N}^\mathbb{N} \), the following are equivalent for all \( i \in \mathbb{N}^+ \).

(a) \( \alpha_i > 0 \).
(b) \( B_i^\xi \neq \emptyset \).
(c) \( B_i^\xi \) is infinite.
(d) \( \lim_{n \to \infty} \frac{|\{ \ell \in n \mid \xi(\ell) = i \}|}{n} = \alpha_i > 0 \).

In this case, we say that \( \xi \) is \( \mu_\alpha \)-random.
Example

Suppose that $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$. Let $\xi$ be $\mu_\alpha$-generic. Then there are the following obvious possibilities for a corresponding random subgroup.

(i) $H_\xi = \text{Fin}(B_1^\xi) \times \text{Fin}(B_2^\xi)$.

(ii) $H_\xi = \text{Alt}(B_1^\xi) \times \text{Alt}(B_2^\xi)$.

(iii) $H_\xi = \text{Fin}(B_1^\xi) \times \text{Alt}(B_2^\xi)$.

(iv) $H_\xi = \text{Alt}(B_1^\xi) \times \text{Fin}(B_2^\xi)$.

(v) $H_\xi = \{ (\pi, \theta) \in \text{Fin}(B_1^\xi) \times \text{Fin}(B_2^\xi) \mid \text{sgn}(\pi) = \text{sgn}(\theta) \}$. 
The ergodic IRSs of $\text{Fin}(\mathbb{N})$

- Let $I = \{ i \in \mathbb{N}^+ \mid \alpha_i > 0 \}$ and let $S_{\alpha} = \bigoplus_{i \in I} C_i$, where each $C_i = \{ \pm 1 \}$ is cyclic of order 2.

- Fix some subgroup $A \leq S_{\alpha}$.

- Let $\xi$ be $\mu_{\alpha}$-random and let $s_\xi$ be the homomorphism

$$s_\xi : \bigoplus_{i \in I} \text{Fin}(B_i^\xi) \to \bigoplus_{i \in I} C_i$$

$$(\pi_i) \mapsto (\text{sgn}(\pi_i)).$$

- Let $\xi \mapsto H_\xi = s_\xi^{-1}(A)$.

- Then $\nu^A_{\alpha} = (f^A)_* \mu_{\alpha}$ is an ergodic IRS of $\text{Fin}(\mathbb{N})$. 
The ergodic IRSs of $\text{Fin}(\mathbb{N})$}

**Theorem (Vershik 2011 with corrections by Thomas 2013)**

If $\nu$ is an ergodic IRS of $\text{Fin}(\mathbb{N})$, then there exists $\alpha$, $A$ as above such that $\nu = \nu^A_{\alpha}$.

- The proof makes use of:

**Theorem (Wielandt 1959)**

If $H \leq \text{Fin}(\mathbb{N})$ is a primitive subgroup, then $H = \text{Alt}(\mathbb{N})$, $\text{Fin}(\mathbb{N})$. 

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The proof begins ...

- Suppose that $\nu$ is an ergodic IRS of $\text{Fin}(\mathbb{N})$.
- For each $H \in \text{Sub}_{\text{Fin}(\mathbb{N})}$, let $p(H)$ be the partition of $\mathbb{N}$ into $H$-orbits.
- Then $m = p_* \nu$ is an ergodic random invariant partition of $\mathbb{N}$.
- Hence there exists $\alpha \in [0, 1]^\mathbb{N}$ such that $m = \varphi_* \mu_\alpha$.
- Let $I = \{ i \in \mathbb{N}^+ \mid \alpha_i > 0 \}$.
- Then for $\nu$-a.e. $H \in \text{Sub}_{\text{Fin}(\mathbb{N})}$, there exists a $\mu_\alpha$-generic $\xi \in \mathbb{N}^\mathbb{N}$ such that $p(H)$ is
  \[ \mathbb{N} = \bigcup_{i \in I} B_i^{\xi} \cup \bigcup_{n \in B_0^{\xi}} \{ n \}. \]
An application of Wielandt’s Theorem

Lemma

For \( \nu \)-a.e. \( H \in \text{Sub}_{\text{Fin}(\mathbb{N})} \), for all \( i \in I \), the subgroup \( H \) induces at least \( \text{Alt}(B_i^\xi) \) on \( B_i^\xi \).

- If not, then there exists a fixed \( i \in I \) such that for \( \nu \)-a.e. \( H \in \text{Sub}_{\text{Fin}(\mathbb{N})} \), the subgroup \( H \) preserves a nontrivial equivalence relation \( E_B \) on \( B = B_i^\xi \).
- Clearly each \( E_B \)-class is finite.

Claim

We can choose \( E_B \) in a \( \text{Fin}(\mathbb{N}) \)-equivariant Borel manner.
The Borel map $H \mapsto E_H = E_B \cup \text{Id}(\mathbb{N} \setminus B)$ is $\text{Fin}(\mathbb{N})$-equivariant.

Thus $m' = e_* \nu$ is an ergodic random invariant partition which concentrates on the equivalence relations $E$ such that
- every $E$-class is finite;
- there exists an $E$-class of some fixed size $k > 1$.

For each $S \in [\mathbb{N}]^k$, let $C_S$ be the event that $S$ is an $E$-class.

Then there exists a fixed $r > 0$ such that $m'(C_S) = r$ for all $S \in [\mathbb{N}]^k$.

Since the events $\{ C_S \mid 0 \in S \in [\mathbb{N}]^k \}$ are mutually exclusive, this is a contradiction.
Towards a proof of the Claim

Lemma

If $H \leq \text{Fin}(\mathbb{N})$ acts transitively but imprimitively on $\mathbb{N}$, then there exist only finitely many minimal nontrivial $H$-invariant equivalence relations.

- Suppose that $\{ E_n \mid n \in \mathbb{N} \}$ are distinct minimal $H$-invariant nontrivial equivalence relations.
- Let $C_n$ be the $E_n$-class such that $0 \in C_n$.
- If $n \neq m$, then $C_n \cap C_m = \{ 0 \}$.
- Choose $a_n \in C_n \setminus \{ 0 \}$.
- Then there exists $\pi \in H$ such that $\pi(0) = a_0$.
- If $n > 0$, then $a_0 \in \pi(C_n) \neq C_n$ and so $\pi(a_n) \neq a_n$.
- But this means that $\pi \notin \text{Fin}(\mathbb{N})$. 
Proof of the Claim

Theorem (Neumann 1975 & Segal 1974)

If \( H \leq \text{Fin}(\mathbb{N}) \) acts transitively but imprimitively on \( \mathbb{N} \), then either:

(i) there exists a unique maximal nontrivial \( H \)-invariant equivalence relation \( E_{\text{max}} \); or

(ii) there exists a sequence of nontrivial \( H \)-invariant equivalence relations \( R_0 \subset R_1 \subset \cdots \subset R_\ell \subset \cdots \) such that \( \mathbb{N}^2 = \bigcup R_\ell \).

- If (i) holds, we can choose \( E_{\text{max}} \).
- Suppose that (ii) holds and let \( E_1, \cdots, E_n \) be the minimal nontrivial \( H \)-invariant equivalence relations.
- Then there exists a unique nontrivial \( H \)-invariant equivalence relation \( R \) which is minimal such that \( E_1 \cup \cdots \cup E_n \subseteq R \).
The analysis continues ...

Lemma

For $\nu$-a.e. $H \in \text{Sub}_{\text{Fin}(\mathbb{N})}$, for all $i \in I$, the subgroup $H$ induces at least $\text{Alt}(B_i^\xi)$ on $B_i^\xi$.

Lemma

For $\nu$-a.e. $H \in \text{Sub}_{\text{Fin}(\mathbb{N})}$, we have that $\bigoplus_{i \in I} \text{Alt}(B_i^\xi) \leq H$. 
Proposition

There exists a partition \( \{ F_\alpha \mid \alpha \in \Omega \} \) of \( I \) into finite pieces such that

\[
H \cap \bigoplus_{i \in I} \text{Alt}(B_i^\xi) = \bigoplus_{\alpha \in \Omega} \text{Diag} \left( \bigoplus_{j \in F_\alpha} \text{Alt}(B_j^\xi) \right),
\]

where the diagonal subgroups are determined by unique bijections \( T_{jk} : B_j^\xi \to B_k^\xi \) for \( j, k \in F_\alpha \).

Let \( E \) be the ergodic invariant random partition defined by

\[
 n E m \iff \left( \exists j, k \right) T_{jk}(n) = m.
\]

Since \( E \) has finite classes, it follows that each \( |F_\alpha| = 1 \).
Remark
An easy ergodicity argument shows that there exists a fixed subgroup $A \leq \bigoplus_{i \in I} C_i$ such that $\nu$ concentrates on the same collection $X^A_\alpha \subseteq \text{Sub}_G$ of subgroups as $\nu^A_\alpha$.

Question
But why is $\nu = \nu^A_\alpha$?

Theorem
The action $\text{Fin}(\mathbb{N}) \curvearrowright X^A_\alpha$ is uniquely ergodic.
The Pointwise Ergodic Theorem

**Theorem (Vershik 1974 & Lindenstrauss 1999)**

Suppose that $G = \bigcup_{n \in \mathbb{N}} G_n$ is the union of an increasing chain of finite groups and that $G \varsubsetneq (Z, \mu)$ is ergodic. Then for $\mu$-a.e. $z \in Z$, for every $\mu$-measurable subset $Y \subseteq Z$,

$$
\mu(Y) = \lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z \in Y \}|.
$$

**Remark**

Thus, to prove that $G \varsubsetneq (Z, \mu)$ is uniquely ergodic, it is enough to show that if $z, z' \in Z$ and $U \subseteq Z$ is a basic open subset, then

$$
\lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z \in U \}| = \lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z' \in U \}|.
$$

The End