The modular forms of the simplest quantum field theory

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Thursday, February 15, 2018

Abstract

Much of the $(2,5)$ minimal model in conformal field theory is described by very classical mathematics: Schwarz' work on algebraic hypergeometric functions, Klein's work on the icosahedron, the Rogers-Ramanujan functions etc. Unexplored directions promise equally beautiful results.

1 The $(2,5)$ MM for $g = 1$

1.1 Some ODEs

For $g = 1$, the 1-point function $\langle T(z)\rangle$ is constant in position, denoted by $\langle \mathbf{T}\rangle$. For the $(2,5)$ minimal model, the Virasoro OPE is given by

$$T(z) \otimes T(0) \mapsto \frac{c/2}{z^4}.1 + \frac{1}{z^2}\{T(z) + T(0)\} - \frac{1}{5}T''(0) + O(z)\,,$$

where $c = -22/5$. This implies the 2-point function

$$\langle T(z)T(0)\rangle = \frac{c}{2}\wp^2(z|\tau)\langle \mathbf{1}\rangle + 2\wp(z|\tau)\langle \mathbf{T}\rangle - c\frac{\pi^4}{15}E_4\langle \mathbf{1}\rangle\,.$$

Changes in the modulus τ are generated by the Virasoro field T. We obtain a system of ODEs of the type studied by [8] and [3]:

$$\frac{1}{2\pi i}\frac{d}{d\tau}\langle \mathbf{1}\rangle = \oint \langle T(z)\rangle \frac{dz}{(2\pi i)^2} = \frac{1}{(2\pi i)^2}\langle \mathbf{T}\rangle$$
$$\frac{1}{2\pi i}\frac{d}{d\tau}\langle \mathbf{T}\rangle = \oint \langle T(w)T(z)\rangle \frac{dz}{(2\pi i\,)^2} = \frac{1}{6}E_2\langle \mathbf{T}\rangle + \frac{11}{3600}(2i\pi)^2 E_4\langle \mathbf{1}\rangle\,,$$

or in terms of the Serre derivative $\mathfrak{D} := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{6}E_2$ (defined on modular forms of weight $2k$),

$$\mathfrak{D}^2\langle \mathbf{1}\rangle = \frac{11}{3600}\,E_4\langle \mathbf{1}\rangle\,. \quad (1)$$

Its solutions are the 0-point functions named after Rogers-Ramanujan (in the following referred to as RR)

$$\langle \mathbf{1}\rangle_i = q^{-\frac{c}{24}}\chi_i\,, \quad i = 1, 2\,,$$

for the characters

$$\chi_1 = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + \dots \quad \text{(vacuum)}$$

$$\chi_2 = q^{-\frac{1}{5}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = q^{-\frac{1}{5}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + \dots\right) .$$

Here $q = \exp(2\pi i\tau)$. The partition function is

$$Z = |\langle \mathbf{1} \rangle_1|^2 + |\langle \mathbf{1} \rangle_2|^2 .$$

The space of all fields factorises as

$$F = F_V \otimes \overline{F_V} \oplus F_W \otimes \overline{F_W} ,$$

where F_V and F_W denote the space of holomorphic fields (irreps of the Virasoro algebra) that correspond to states in V and W, respectively, and the bar marks complex conjugation.

We shall use the algebraic description of the torus as a double cover $\mathbb{P}^1_{\mathbb{C}}$ defined by

$$y^2 = x(x-1)(x-\lambda) ,$$

where $\lambda \in \mathbb{C}$ is the squared Jacobi modulus. Describing the change of λ by an action of $T(x)$, we find the ODE

$$\frac{d^2}{d\lambda^2} f + p \frac{d}{d\lambda} f + q f = 0 \tag{2}$$

with rational coefficients

$$q = \frac{-\alpha\beta}{\lambda(1-\lambda)} , \quad p = \frac{\gamma}{\lambda} + \frac{\gamma - (\alpha+\beta+1)}{1-\lambda} ,$$

where

$$(\alpha,\beta;\gamma) = \left(\frac{7}{10}, \frac{11}{10}; \frac{7}{5}\right) \quad \text{or} \quad \left(\frac{3}{10}, -\frac{1}{10}; \frac{3}{5}\right) .$$

Its equivalence to (1) can be seen from [5]

$$\frac{2\lambda - 1}{\lambda(\lambda-1)} d\lambda = \pi i\, E_2 d\tau - 6d(\log \ell) ,$$

where ℓ is the inverse length of the real period.

Comparison of the two approaches yields

$$\langle \mathbf{1} \rangle_1 = [\lambda(\lambda-1)]^{-c/24}\, {}_2F_1\left(\frac{7}{10}, \frac{11}{10}; \frac{7}{5}; \lambda\right)$$

$$\langle \mathbf{1} \rangle_2 = [\lambda(\lambda-1)]^{-1/5-c/24}\, {}_2F_1\left(\frac{3}{10}, -\frac{1}{10}; \frac{3}{5}; \lambda\right) .$$

These relations seem to be new though they're closely related to Schwarz' work [10] as will be indicated in the rest of this section.

2 Algebraicity of the Rogers-Ramanujan characters

Besides the analytic approach, there is an algebraic approach to the characters.

2.1 Motivation

From general theory, we know that any two modular functions are algebraically dependent [13, Propos 3, p. 12], so

Propos. 1. *The Rogers Ramanujan functions are algebraic in the j-invariant.*

We are interested in generalising this result to higher genus. As a preparation, the specific algebraic equations for the Rogers-Ramanujan functions will be discussed.

2.2 Schwarz' list

A necessary condition for the general solution of the hypergeometric differential equation (2) to be algebraic in λ is that $\alpha, \beta, \gamma \in \mathbb{Q}$ (Kummer), which we will assume in the following.

Propos. 2. *Let f_1, f_2 be solutions of (2), for some choice of $\alpha, \beta, \gamma \in \mathbb{Q}$, such that*

$$s = f_1 / f_2$$

is algebraic. Then f_1, f_2 are themselves algebraic.

A particularly neat argument is due to Heine [10, and reference therein].

Proof. Let $W = f_1' f_2 - f_2' f_1$ be the Wronskian. Since s is algebraic and $s' = W/f_2^2$, it suffices to show that W is algebraic: We have

$$W' = f_1'' f_2 - f_2'' f_1 = -pW$$

by eq. (2), so for A, B such that

$$p(\lambda) = -\frac{A}{\lambda} - \frac{B}{\lambda - 1} ,$$

we have

$$W \sim \exp\left(-\int p \, d\lambda\right) = \lambda^A (\lambda - 1)^B .$$

By assumption $A, B \in \mathbb{Q}$. □

Given two independent algebraic solutions f_1, f_2 to (2), their quotient

$$s = f_1 / f_2$$

solves a third order differential equation in λ [10, p. 299], which involves the Schwarzian derivative. By linearity of (2), the space of solutions is invariant under Möbius transformations. s defines a map

$$\begin{aligned} s : \quad \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\} &\to \mathbb{P}^1_{\mathbb{C}} \left(\cong S^2\right) \\ \lambda &\mapsto (f_1 : f_2) . \end{aligned}$$

Suppose $f_1^{\mathbb{R}}, f_2^{\mathbb{R}}$ are real on $(0,1)$ and their quotient $s^{\mathbb{R}} = f_1^{\mathbb{R}}/f_2^{\mathbb{R}}$ maps the interval $(0,1)$ onto a segment $I_{(0,1)}$ of $\mathbb{P}^1_{\mathbb{R}} \cong S^1$. Via an analytic extension to $\mathfrak{h}$, $s^{\mathbb{R}}$ can be extended to the intervals $(-\infty, 0)$ and $(1, \infty)$. For $\varepsilon > 0$, the interval $(-\varepsilon, \varepsilon)$ is mapped to two arcs forming some angle. Together, the images of $(0,1)$, $(1,\infty)$ and $(-\infty, 0)$ form a triangle in $\mathbb{P}^1_{\mathbb{C}}$. In the elliptic case (angular sum $> 180°$), the triangle is conformally equivalent to a spherical triangle on S^2 whose edges are formed by arcs of great circles.

By crossing any of the intervals $(1,\infty)$ $(0,1)$, or $(-\infty,0)$, $s^{\mathbb{R}}$ can be further continued to $\mathbb{H}^-$. We have a correspondence between reflection symmetry w.r.t. the real line in the λ-plane and circle inversion w.r.t. the respective triangle edge in $\mathbb{P}^1_{\mathbb{C}}$.

Analytic continuating along paths circling the singularities in any order may in general produce an infinite number of triangles in $\mathbb{P}^1_{\mathbb{C}}$. The number is finite iff the quotient of solutions is finite [9, Sect. 20]. For angle sums $\leq 180°$ finite coverings are impossible.

Thus the problem is transformed into sorting out all spherical triangles whose symmetric and congruent repetitions lead to a finite number only of triangles of different shape and position.

A necessary condition for a spherical shape and its symmetric and congruent repetions to form a closed Riemann surface is that the edges lie in planes which are symmetry planes of a regular polytope.

For the spherical triangles, this leads to a finite list of triples of angles that correspond to platonic solids.

The Rogers-Ramanujan functions feature as the most symmetric case (no. XI, i.e. all three angles equal $2\pi/5$) in the list of Schwarz [10].

The compactified fundamental domain $\overline{\Gamma_1 \setminus \mathfrak{h}} = \Gamma_1 \setminus (\mathfrak{h} \cup \mathbb{Q} \cup \{\infty\})$ of Γ_1 [13] is conformally equivalent to $\mathbb{P}^1_{\mathbb{C}}$. The j-invariant defines a Hauptmodul for Γ_1.

On the other hand, the modular curve of the principal congruence subgroup $\Gamma(N)$ has $g = 0$ iff $1 \leq N \leq 5$. For $N \geq 2$, the map $\Gamma(N) \setminus \mathfrak{h} \to \mathbb{P}^1_{\mathbb{C}}$ is conformal outside of the cusps. Thus the angle $\frac{\pi}{3}$ at $\rho = \frac{1}{2}(1 + i\sqrt{3})$ is preserved under this map. Since N copies of the fundamental domain of Γ_1 meet in the cusp at $i\infty$, the compatified fundamental domain of $\Gamma(N)$ defines a finite covering iff $\frac{2\pi}{N} + \frac{2\pi}{3} > \pi$, or equivalently $N < 6$.

For $N = 5$, the angle at the image of $i\infty$ equals 72°. The modular curve $\Gamma(5) \setminus (\mathfrak{h} \cup \mathbb{Q} \cup \{\infty\})$ has the symmetry of an icosahedron. By modularity on $\Gamma(5)$, $r(\tau) = \langle \mathbf{1} \rangle_1 / \langle \mathbf{1} \rangle_2$ defines a map

$$\Gamma(5) \setminus \mathfrak{h} \to \mathbb{P}^1_{\mathbb{C}}$$

$r(\tau)$ is a Hauptmodul for $\Gamma(5)$. We have $[\Gamma_1 : \Gamma(5)] = 120$ [2], so the fundamental domain of $\Gamma(5)$ defines an 120-fold covering of $\mathbb{P}^1_{\mathbb{C}}$, and r and j are rational functions of one another.

2.3 Klein's invariants

Felix Klein reverses the order of arguments used by Schwarz.

Theorem 1. *(Felix Klein) The icosahedral irrationality*

$$q^{1/5}\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}}}\,, \tag{3}$$

[4, Part I, eq. (20), p. 146] is algebraic.

For a proof, consider an icosahedron inscribed in the unit sphere. Subdivide each face into 6 triangles by connecting its centroid with the surrounding vertices and edge midpoints. Projecting the vertices and edges onto the sphere $\mathbb{P}^1_{\mathbb{C}}$ from its center results in a tessellation of the sphere by triangles. Let $\tilde{V}$ be the invariant homogeneous poynomial in projective coordinates s_1, s_2 on $\mathbb{P}^1_{\mathbb{C}}$ with $\deg \tilde{V} = 12$, which has a simple root at $s_1 s_2 = 0$. (We assume here that north and south pole are vertices). The Hessian form $\tilde{F}$ of $\tilde{V}$ and the functional determinant $\tilde{E}$ of $\tilde{V}$ and $\tilde{F}$,

$$\tilde{F} = |(\partial_{ij}\tilde{V})|\,, \quad \tilde{E} = |(\partial_i\tilde{V}, \partial_j\tilde{F})|$$

have roots corresponding to $2\cdot 10 = 20$ face centers and to $11 + 19 = 30$ mid-edge points of the icosahedron, respectively. Thus $\deg\tilde{F} = 20$ and $\deg\tilde{E} = 30$. $\tilde{V}$, $\tilde{F}$ and $\tilde{E}$ satisfy a syzygy [1].

By stereographic projection, the tessellation of the sphere gives rise to a configuration of intersecting arcs on the extended complex plane $\hat{\mathbb{C}}$ (with coordinate z), with[1] 11 finite vertices, 20 face centers, and 30 edge points which define simple roots of monic polynomials V and F and E, respectively.

To the icosahedral symmetry group A_5 corresponds the subgroup $G_{60} \subseteq PSL(2,\mathbb{C})$ of Möbius transformations. Under an action of G_{60}, F and V transform as modular forms of weight -20 and -12, respectively, so that

$$J(z) = -\frac{F^3(z)}{V^5(z)}$$

is invariant. For $\tilde{z} = z^5$, the associated icosahedral equation reads

$$(\tilde{z}^4 - 228\tilde{z}^3 + 494\tilde{z}^2 + 228\tilde{z} + 1)^3 + \tilde{z}(\tilde{z}^2 + 11\tilde{z} - 1)^5 J(z) = 0\,.$$

The icosahedral equation is the minimal polynomial of $\tilde{z}$ over $\mathbb{Q}(J)$. By the Jacobi triple product identity, we have [12]

$$\prod_{n=0,\pm 2 \bmod 5}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}}$$
$$\prod_{n=0,\pm 1 \bmod 5}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}}$$

so the function (3) is $z = r(\tau) = \langle\mathbf{1}\rangle_1/\langle\mathbf{1}\rangle_2$. Moreover, $J(z)$ equals the classical j-invariant [1], expressed as rational function of $r(\tau)$. $r(\tau)^5$ defines a 12-fold covering of $\mathbb{P}^1_{\mathbb{C}}$.

[1] We call a point of intersection an edge point, a face point or a vertex depending on its origin in the icosahedron.

r is determined up to linear fractional transformation, so its Schwarzian derivative is unique, and we obtain a third order ODE for r as a function of j. $\langle\mathbf{1}\rangle_1$ and $\langle\mathbf{1}\rangle_2$ define projective coordinates or elements in the two-dimensional space of global rational sections in the sheaf $O(1)$. Thus they define two solutions of a linear 2nd order ODE in j with rational coefficients.

As an algebraic function of

$$j = 2^8 \frac{(1-\lambda(1-\lambda))^3}{\lambda^2(1-\lambda)^2} ,$$

r is also an algebraic function of λ. This leads to our corresponding ODEs w.r.t λ.

3 The $(2,5)$ MM for $g \geq 2$

3.1 ODEs for the 0-point functions

Theorem 2. *[6]*
0-point functions for genus 2 solve a 5th order linear ODE (w.r.t. any of its ramification points) with regular singularities. (The system is given explicitely.)

The main idea of the proof is to use algebraic coordinates, $x = \wp(z|\tau)$, $y = \partial_z\wp(z|\tau)$.

Alternatively: Use the rich theory of elliptic functions by letting the genus $g = 2$ surface degenerate (Deligne-Mumford compactification (1969) of the moduli space of Riemann surfaces).

For $i = 1, 2$, let (Σ_i, P_i) with $P_i \in \Sigma_i$ be a non-singular Riemann surface of genus g_i with puncture P_i. Let z_i be a local coordinate vanishing at P_i. Excise sufficiently small discs $\{|z_1| < \varepsilon\}$ and $\{|z_2| < \varepsilon\}$ from Σ_1 and Σ_2, respectively, and sew the two remaining surfaces by the condition

$$z_1 z_2 = \varepsilon^2 \tag{4}$$

on tubular neigbourhoods of the circles $\{|z_i| = \varepsilon\}$.

This operation yields a non-singular Riemann surface of genus $g_1 + g_2$ with no punctures. The $g = 2$ partition function is obtained perturbatively as a power expansion in ε. Two possibilities [11]:

1. The $g = 2$ surface decomposes into **two tori** (with modulus τ and $\hat{\tau}$, respectively) when three ramification points run together (cutting through the neck along a cycle that is homologous to zero). We have for $a, b \in \{1, 2\}$ (Gilroy & Tuite, and [7]),

$$\langle 1\rangle^{g=2}_{a,b}(q,\hat{q},\varepsilon) = \langle 1\rangle_a \widehat{\langle 1\rangle}_b - \frac{2}{c}\varepsilon^2 \langle T\rangle_a \widehat{\langle T\rangle}_b - \frac{7}{31c}\varepsilon^6 \langle L_4 L_2 1\rangle_a \widehat{\langle L_4 L_2 1\rangle}_b + O(\varepsilon^8) \,.$$

   (The hat refers to the modulus $\hat{\tau}$.) In addition there is a 5th solution:

$$\langle 1\rangle^{g=2}_{\varphi}(q,\hat{q},\varepsilon) = \varepsilon^{-1/5} \left(\eta\,\widehat{\eta}\right)^{-2/5} \left\{1 + \frac{13}{8,208,000}(2\pi i)^8 \varepsilon^4 E_4 \widehat{E}_4 + O(\varepsilon^6)\right\} .$$

2. One obtains a **single torus** by letting two ramification points run together (cutting through a genus $g = 2$ surface along a cycle that is not homologous to zero) [7].

The invariant is

$$Z^{g=2} = \sum_{i}^{4} |\langle \mathbf{1} \rangle_i^{g=2}|^2 + \kappa |\langle \mathbf{1} \rangle_5^{g=2}|^2 ,$$

for some $\kappa \in \mathbb{R}$ (so that Z is modular).

3.2 Related ODEs

The physical interpretation of the 5th solution requires another field Φ with (locally) $\Phi = \varphi_{\text{hol}} \otimes \overline{\varphi}_{\text{hol}}$ where $\varphi := \varphi_{\text{hol}}$ is lowest weight vector (of weight $-1/5$) in the irrep F_W of the Virasoro algebra. Its 1-point function satisfies

$$\mathcal{D}\langle \varphi \rangle = 0 .$$

Thus

$$\langle \varphi \rangle = \eta^{-2/5} = q^{-\frac{1}{60}} \prod_{n \geq 1} (1 - q^n)^{-2/5}$$

We apply the methods used previoulsy in F_V to $\varphi \in F_W$. The continuation of the theory from $g = 1$ to $g = 2$ requires both lowest weight vectors.

Claim 1. *The* 2*-pt function of* φ *satisfies a* 3*rd order ODE with regular singularities,*

$$\frac{25}{12} \langle \varphi^{(3)}(z) \varphi(0) \rangle = h \wp'(z) \langle \varphi(z) \varphi(0) \rangle + \wp(z) \langle \varphi'(z) \varphi(0) \rangle .$$

In algebraic coordinates (and the corresponding field $\check{\varphi}$*), the above ODE reads*

$$y^{4/5} \left(p(x) \frac{d^3}{dx^3} + f(x) \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right) \Psi(x) = 0 ,$$

where $\Psi(x) = \langle \check{\varphi}(x) \varphi(0) \rangle$*,*

$$p(x) = 4 \left(x^3 - \frac{\pi^4}{3} E_4 x - \frac{2}{27} \pi^6 E_6 \right) ,$$

and

$$\begin{aligned} f &= \frac{6}{5} p' \\ g &= \frac{3}{100} \frac{[p']^2}{p} + \frac{9}{50} p'' \\ h &= -\frac{33}{500} \frac{[p']^3}{p^2} + \frac{33}{250} \frac{p' p''}{p} - \frac{288}{125} . \end{aligned}$$

In particular, the ODE has simple poles at the four ramification points.

A proof of the statement can be found in [7].

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