Rooted Tree Maps

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1712.01601 } joint works with H. Bachmann
1801.05381 }

§1. A review of Connes-Kreimer Hopf algebra of rooted trees
§2. Rooted tree maps
§3. Applications to MZV's
3.1 , 3.2 , 3.3
§4. Problems

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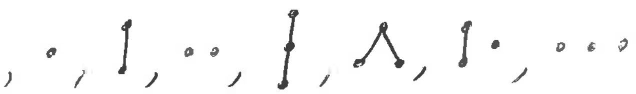
§1 **Def.** - A rooted tree is a (non-empty) connected finite graph with no loops and has a special vertex called the root such that any edge is oriented away from it.

- A planar rooted tree is a rooted tree t together with ordering of outgoing edges for each vertex of t.

- We just consider non-planar rooted trees.
  i.e. Do not distinguish  root and  , for example.

- A product of rooted trees is given by disjoint union and called a forest.

e.g. $\mathbb{I}$ (empty forest) ,  , etc.

Let $H := \sum\limits_{f:\text{forest}} \mathbb{Q} f$: free comm. $\mathbb{Q}$-alg.

e.g.  $\in H$.

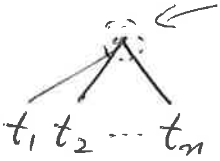
$\mathcal{T}$:= the set of all rooted trees

$\langle \mathcal{T} \rangle_{\mathbb{Q}}$:= its linear span over $\mathbb{Q}$

Define the $\mathbb{Q}$-linear map $B_+ : H \longrightarrow \langle \mathcal{T} \rangle_{\mathbb{Q}}$ by

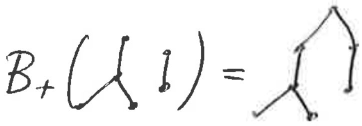
$B_+(\mathbb{I}) = \cdot$

new root

$B_+(\underbrace{t_1 \cdots t_n}_{\text{forest}}) =$  (single tree)

t_j : tree

grafting operator

e.g. 

Fact $\forall t \in \mathcal{T}$, $\exists ! f$: forest s.t. $t = B_+(f)$.

<u>Thm</u> (Connes-Kreimer, '98) H is a Hopf algebra

by

coproduct $\Delta : H \longrightarrow H \otimes H$ s.t.

$$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$$

$$\Delta(fg) = \Delta(f)\Delta(g)$$

$$\Delta(t) = t \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta(f)$$

for $\underset{(\text{tree})}{t} = B_+(f)$,

counit "$\hat{\mathbb{I}}$", and antipode "S".

(omit to define)

<u>e.g.</u> $\Delta(\bullet) = \Delta(B_+(\mathbb{I})) = \bullet \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \underbrace{\Delta(\mathbb{I})}_{\mathbb{I} \otimes \mathbb{I}}$

$$= \bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet$$

$$\Delta(\bullet\,\bullet) = \Delta(\bullet)^2 = \bullet\,\bullet \otimes \mathbb{I} + 2\,\bullet \otimes \bullet + \mathbb{I} \otimes \bullet\,\bullet$$

$$\Delta(\wedge) = \wedge \otimes \mathbb{I} + \bullet\,\bullet \otimes \bullet + 2\,\bullet \otimes \vert + \mathbb{I} \otimes \wedge$$

<u>Fact</u> Δ is coassociative, i.e.

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$$

§2 Notation

$$\mathfrak{H} := \mathbb{Q}\langle x, y\rangle \supset \mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y$$

$$\mathfrak{H} \ni z := x + y$$

$$M : \mathfrak{H} \otimes \mathfrak{H} \longrightarrow \mathfrak{H}, \quad M(v \otimes w) = vw$$

$$R_u(w) := wu \qquad (u \in \mathfrak{H})$$

First we regard $\mathbb{I}$ as the identity map on $\mathfrak{H}$.

<u>Thm 1</u> (T.) For any forest $f\ (\neq \mathbb{I})$, we can define the $\mathbb{Q}$-linear map from $\mathfrak{H}$ to $\mathfrak{H}$, which is also denoted by f, by

(i) $f = \bullet \Rightarrow f(x) = -f(y) = xy$

(i') $(B_+(f))(u) := R_y R_{y+z} R_y^{-1} f(u)$ for $u \in \{x, y\}$

(i'') $f = gh\ (g, h \neq \mathbb{I}) \Rightarrow f(u) := g(h(u))$ for $u \in \{x, y\}$

(ii) $f(wu) := M((\Delta(f))(w \otimes u))$ for $w \in \mathfrak{H}$, $u \in \{x, y\}$.

e.g. $\Lambda(x) = (B_+(\bullet\,\bullet))(x) = R_y R_{y+z} R_y^{-1} \underbrace{\bullet\,\bullet(x)}_{=\ \bullet(xy)\ =\ \bullet(x)y + x\bullet(y)\ =\ xy^2 - x^2y} = -x^3y - 2x^2y^2 + xyxy + 2xy^3$

$$\Lambda(xy) = \Lambda(x)y + \bullet\,\bullet(x)\bullet(y) + 2\bullet(x)\underbrace{\mathrm{I}(y)}_{=\ (B_+(\bullet))(y)\ =\ R_y R_{y+z} R_y^{-1}\bullet(y)\ =\ \cdots} + x\underbrace{\Lambda(y)}_{=\ -\Lambda(x)}$$

$$= x^4y + x^3y^2 - 4x^2y^3 - 2xyx^2y - 3xyxy^2 - xy^2xy + 2xy^4.$$

Put $\psi_f := [f, R_x] = fR_x - R_x f$ for a forest map f.

Thm 2 (T.) For any forest maps f and g, we have

a) $\exists \phi_f$ s.t. $\psi_f = R_y \phi_f R_x$

b) $f(\mathbb{Q}x + \mathbb{Q}y + \mathfrak{h}^0) \subset x\mathfrak{h}y$

c) $\phi_{B_+(f)} = f + R_z \phi_f$

d) $[f, g] = 0$, $[f, R_z] = 0$

e) $\phi_f \in \mathbb{Q}[R_z, \mathcal{T}]_{(\deg f - 1)}$

f) $f(vw) = M((\Delta(f))(v \otimes w))$, $\forall v, w \in \mathfrak{h}$.

Here, $\mathbb{Q}[X]_{(d)}$ denotes the degree d homogenous part of $\mathbb{Q}[X]$, $\deg(R_z) := 1$, and $\deg(f) := \#$ vertices of f.

Rmk: The property f) means that the recursive rule of f does not depend on how to deconcatenate a word.

Example of ϕ_f $\quad \psi_{\mathbb{I}} = [\mathbb{I}, R_x] = 0$, $\therefore \phi_{\mathbb{I}} = 0$.

$\psi_\bullet = [\bullet, R_x] = \bullet R_x - R_x \bullet$

$$\psi_\bullet(w) = \underbrace{\bullet(wx)}_{\substack{\| \\ \bullet(w)x + w\bullet(x)}} - \bullet(w)x = w\bullet(x) = wxy$$

$\therefore \psi_\bullet = R_y \phi_\bullet R_x$, where $\phi_\bullet = id$.

$\phi_{\rceil} = \bullet + R_z$, $\phi_{\bullet\bullet} = 2\bullet - R_z$, $\phi_{\rfloor} = \rceil + R_z + R_z^2$,

$\phi_{\wedge} = \bullet\bullet + 2R_z\bullet - R_z^2$, $\phi_{\rceil\bullet} = \bullet\bullet + \rceil - R_z^2$, etc.

Fact $\phi_{fg} = f\phi_g + \phi_f g - \phi_f R_z \phi_g$ holds for any forest maps f and g.

§3 **3.1** Let $Z:\mathfrak{H}^0 \to \mathbb{R}$ be the $\mathbb{Q}$-linear map defined by

$$Z(1)=1,\quad Z(x^{k_1-1}y\cdots x^{k_r-1}y) = \zeta(k_1,\dots,k_r)$$
$$(k_1>1)$$
$$\zeta(k_1,\dots,k_r) = \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \quad : \text{MZV}$$

Thm 3 (T.) For any forest map $f\ (\neq \mathrm{I})$, we have $f(\mathfrak{H}^0) \subset \ker Z$.

e.g. $\Lambda(xy) \overset{Z}{\longmapsto} \zeta(5)+\zeta(4,1)-4\zeta(3,1,1)-2\zeta(2,3)-3\zeta(2,2,1)$
$$-\zeta(2,1,2)+2\zeta(2,1,1,1) = 0$$

3.2 For $n\geq 1$, let $\partial_n : \mathfrak{H} \to \mathfrak{H}$: derivation s.t.
$$\partial_n(x) = -\partial_n(y) = xz^{n-1}y.$$

Thm (Ihara-Kaneko-Zagier, '06) $\partial_n(\mathfrak{H}^0) \subset \ker Z$, $\forall n\geq 1$.

"derivation relation"

Put $\lambda_m :=$ $\left.\begin{matrix}\bullet\\ |\\ \vdots\\ |\\ \bullet\end{matrix}\right\}$ m vertices ("ladder map")

Thm 4 (Bachmann-T.) $$\partial_n = \frac{n}{2^n-1}\sum_{d=1}^{n}\frac{(-1)^{d+1}}{d}\sum_{\substack{m_1+\cdots+m_d=n\\ m_j\geq 1}} \lambda_{m_1}\cdots\lambda_{m_d}, \quad \forall n\geq 1.$$

e.g. $\partial_1 = \bullet$, $\partial_2 = \frac{1}{3}(2\lambda_2 - \bullet\bullet)$,

$\partial_3 = \frac{1}{7}(3\lambda_3 - 3\lambda_2\bullet + \bullet\bullet\bullet)$, etc.

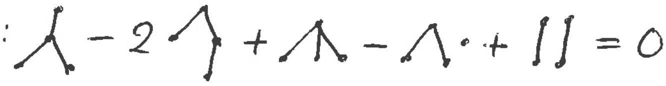
Cor. The derivation relation is implied by the rooted tree maps rel'n.

3.3. **Thm 5** (Bachmann-T.) The "linear part of Kawashima relation" is equivalent to the rooted tree maps relation.

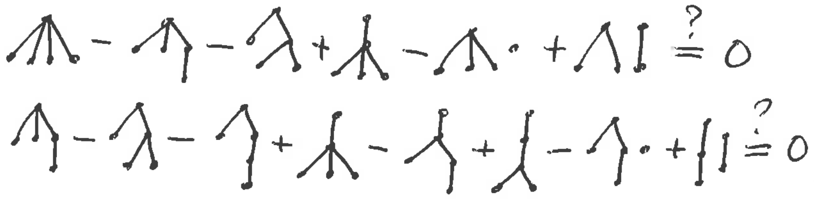
(Details are omitted here.)

§4 - Are there any generalizations of rooted tree maps which induce all relations for MZV's?

- Can we give all relations among rooted tree maps?

e.g. deg 4: 

deg 5: $\bullet$ (LHS of degree 4 rel'n) $= 0$
B_+(LHS of degree 4 rel'n) $= 0$ } "old" (coming from lower degrees)

 } "new"

- Are there any nice explanations of rooted tree maps relations in terms of periods or amplitudes?

⋮

and so forth.