Predicative proof theory of PDL

L. Gordeev

Uni-Tübingen

Bonn, July 3, 2018 (revised)

§1. Abstract

- Propositional dynamic logic **PDL** is presented in Schütte-style mode as one-sided semiformal tree-like sequent calculus $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ with standard cut rule and the omega-rule with principal formulas $[P^{*}]A$. The omega-rule free derivations in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ are finite (trees), while sequents deducible by these finite derivations are valid in **PDL**.
- Cut elimination theorem for $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ is proved in **PA** extended by transfinite induction up to $\varphi_{\omega}(0) > \varepsilon_0$.
- Hence this predicative extension of **PA** proves that any given $[P^{*}]$-free sequent is valid in **PDL** iff it is deducible in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ by a cut- and omega-rule free derivation.
- By the cutfree subformula property this yields Herbrand-style corollary for $\langle P^{*}\rangle$-sequents as well as PSPACE derivability of star-free sequents containing only atomic programs, which in turn implies EXPTIME vs PSPACE consequences.

§2. Hilbert-style formalism of **PDL**

- Language
  1. Formulas FOR (abbr.: A, B, C, F, G, ...) include F-variables x, y, z, ...) and are closed under $\rightarrow$, $\neg$, and modal operation $F \hookrightarrow [P]\,F$ for $P \in \mathrm{PRO}$.
  2. Programs PRO (abbr.: P, Q, R, S, ...) include P-variables p, q, r, ...) and are closed under ; , $\cup$, and star operation *.
- Axioms

(D1) *Axioms of propositional logic.*

(D2) $[P]\,(A \rightarrow B) \rightarrow ([P]\,A \rightarrow [P]\,B)$

(D3) $[P]\,(A \wedge B) \leftrightarrow ([P]\,A \wedge [P]\,B)$

(D4) $[P;Q]\,A \leftrightarrow [P][Q]\,A$

(D5) $[P \cup Q]\,A \leftrightarrow [P]\,A \wedge [Q]\,A$

(D7) $[P^*]\,A \leftrightarrow A \wedge [P]\,[P^*]\,A$

(D8) $[P^*]\,(A \rightarrow [P]\,A) \rightarrow (A \rightarrow [P^*]\,A)$

- Inferences

(MP) $\dfrac{A \quad A \rightarrow B}{B}$ *and* (G) $\dfrac{A}{[P]\,A}$

§3. Semiformal sequent calculus $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$: Language

- Language of $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ includes *seq-formulas* and *sequents*. Seq-formulas are built up from *literals* x and $\neg x$ by connectives $\vee$ and $\wedge$ and modal operations $[P]$ and $\langle P \rangle$. Seq-negation $\overline{F}$ is defined recursively as follows, for any seq-formula F.

$$\overline{x} := \neg x,\ \overline{\neg x} := x,$$
$$\overline{A \vee B} := \overline{A} \wedge \overline{B},\ \overline{A \wedge B} := \overline{A} \vee \overline{B},$$
$$\overline{\langle P \rangle A} := [P]\overline{A},\ \overline{[P]A} := \langle P \rangle \overline{A},$$
$$\overline{\langle P \cup Q \rangle A} := [P \cup Q]\overline{A},\ \overline{[P \cup Q]A} := \langle P \cup Q \rangle \overline{A},$$
$$\overline{\langle P ; Q \rangle A} := [P ; Q]\overline{A},\ \overline{[P ; Q]A} := \langle P ; Q \rangle \overline{A}.$$

- Formulas are represented as seq-formulas by $\neg F := \overline{F}$, $F \to G := \overline{F} \vee G$ and, conversely, by $F \vee G := \neg F \to G$, $F \wedge G := \neg (F \to \neg G)$, $\langle P \rangle F := \neg [P] \neg F$.

$$(\mathrm{Ax})\quad x, \neg x, \Gamma$$

$$(\vee)\quad \frac{A, B, \Gamma}{A \vee B, \Gamma} \qquad (\wedge)\quad \frac{A, \Gamma \qquad B, \Gamma}{A \wedge B, \Gamma}$$

$$\langle\cup\rangle\quad \frac{\langle P\rangle A, \langle R\rangle A, \Gamma}{\langle P \cup R\rangle A, \Gamma} \qquad [\cup]\quad \frac{[P]A, \Gamma \qquad [R]A, \Gamma}{[P \cup R]A, \Gamma}$$

$$\langle ;\rangle\quad \frac{\langle P\rangle\langle R\rangle A, \Gamma}{\langle P; R\rangle A, \Gamma} \qquad [;]\quad \frac{[P][R]A, \Gamma}{[P; R]A, \Gamma}$$

$$\langle *\rangle\quad \frac{\left\langle \vec{Q} \right\rangle \langle P\rangle^m A, \left\langle \vec{Q} \right\rangle \langle P^*\rangle A, \Gamma}{\left\langle \vec{Q} \right\rangle \langle P^*\rangle A, \Gamma}\ (m \geq 0)$$

$$[*]\quad \frac{\cdots \quad \left[\vec{Q}\right][P]^m A, \Gamma \quad \cdots}{\left[\vec{Q}\right][P^*]A, \Gamma}\ (\forall m \geq 0)$$

$$(\textsc{Gen})\quad \frac{A_1, \cdots, A_n}{(P)_{\chi_1}A_1, \cdots, (P)_{\chi_n}A_n, \Gamma}\ (n > 0)$$
$$\text{if } \sum_{i=1}^{n} \chi_i = 1$$

$$(\textsc{Cut})\quad \frac{C, \Gamma \qquad \overline{C}, \Pi}{\Gamma \cup \Pi}$$

Here $\langle P \rangle^m := \overbrace{\langle P \rangle \cdots \langle P \rangle}^{m\ times}$ and $[P]^m := \overbrace{[P] \cdots [P]}^{m\ times}$.

For any $\chi \in \{0, 1\}$ we let $(P)_\chi := \begin{cases} [P], & \text{if } \chi = 1, \\ \langle P \rangle, & \text{if } \chi = 0. \end{cases}$

For any $\overrightarrow{P} = P_1, \cdots, P_k$ $(k \geq 0)$ and $f : [1, k] \to \{0, 1\}$ we let $\left(\overrightarrow{P}\right)_f := (P_1)_{f(1)} \cdots (P_k)_{f(k)}$. By $\left(\overrightarrow{Q}\right)$, $\left\langle \overrightarrow{Q} \right\rangle$ and $\left[\overrightarrow{Q}\right]$ we abbreviate $\left(\overrightarrow{Q}\right)_f$ for arbitrary f, $f \equiv 0$ and $f \equiv 1$, respectively.

§4. $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$: Soundness and completeness

- We assume that $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ *derivations* ∂ are well-founded. The simplest way to implement this assumption is to supply nodes x with ordinals $ord(x)$ such that ordinals of premises are always smaller that the ones of the conclusions. Having this we let $h(\partial) := ord(root(\partial))$ and call it *the height* of ∂.

Theorem (soundness and completeness)

$\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ *is sound and complete with respect to* **PDL**. *Moreover any* **PDL**-*valid sequent (in particular formula) is derivable in* $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ *using ordinals* $< \omega \cdot 2$.

Proof.

Straightforward. □

Note.
The validity of (GEN) follows from that of $(D1)$, $(D2)$, $(D3)$ and plain generalzation (G), e.g. like this:

$$\dfrac{A_1, A_2, \cdots, A_n \overset{\mathcal{L}}{\equiv} A_1 \vee A_2 \vee \cdots \vee A_n}{[P](A_1 \vee A_2 \vee \cdots \vee A_n) \overset{\mathcal{L}}{\equiv} [P](\neg(A_2 \vee \cdots \vee A_n) \to A_1) \underset{(D2)}{\Rightarrow}} \text{(G)}$$

$$[P](\neg(A_2 \vee \cdots \vee A_n) \to [P]A_1) \overset{\mathcal{L}}{\equiv} [P]A_1 \vee \langle P\rangle(A_2 \vee \cdots \vee A_n)$$

$$\underset{(D1)}{\Rightarrow} [P]A_1 \vee \langle P\rangle(A_2 \vee \cdots \vee A_n) \vee \Gamma \underset{(D3)}{\Rightarrow}$$

$$[P]A_1 \vee \langle P\rangle A_2 \vee \cdots \vee \langle P\rangle A_n \vee \Gamma \overset{\mathcal{L}}{\equiv} [P]A_1, \langle P\rangle A_2, \cdots, \langle P\rangle A_n, \Gamma$$

§5. $\mathrm{SEQ}^{\mathrm{PDL}}_{\omega}$: Cut elimination

Theorem (Predicative cut elimination)

The following is provable in **PA** *extended by transfinite induction up to Veblen-Feferman ordinal* $\varphi_\omega(0) > \varepsilon_0$*. Any sequent derivable in* $\mathrm{SEQ}^{\mathrm{PDL}}_{\omega}$ *is derivable in* $\mathrm{SEQ}^{\mathrm{PDL}}_{\omega}$ *-* (CUT)*. Hence any* **PDL***-valid sequent (formula) is derivable in the cut-free fraction of* $\mathrm{SEQ}^{\mathrm{PDL}}_{\omega}$.

Proof.

Define *ordinal complexity* $\mathfrak{o}(-) < \omega^\omega$ of formulas, programs and sequents ("$\#$" is the symmetric sum).

$$
\begin{aligned}
&\mathfrak{o}(x) = \mathfrak{o}(\neg x) = \mathfrak{o}(p) := 0,\\
&\mathfrak{o}(A \vee B) = \mathfrak{o}(A \wedge B) := \max\{\mathfrak{o}(A), \mathfrak{o}(B)\} + 1,\\
&\mathfrak{o}(P \cup Q) := \max\{\mathfrak{o}(P), \mathfrak{o}(Q)\} + 1,\\
&\mathfrak{o}(P; Q) := \mathfrak{o}(P) \# \mathfrak{o}(Q) + 1,\\
&\mathfrak{o}(P^*) := \mathfrak{o}(P) \cdot \omega, \quad \mathfrak{o}(\langle P \rangle A) = \mathfrak{o}([P]A) := \mathfrak{o}(P) \# \mathfrak{o}(A) + 1,\\
&\mathfrak{o}(\Gamma) := \sum \{\mathfrak{o}(A) : A \in \Gamma\}.
\end{aligned}
$$

□

§5. $\text{SEQ}_{\omega}^{\text{PDL}}$: Predicative cut elimination: Proof sketch

Cut elimination operator $\partial \hookrightarrow \mathcal{E}(\partial)$ satisfying $\deg(\mathcal{E}(\partial)) = 1$ is defined by transfinite recursion on $h(\partial)$ and $\deg(\partial)$, where

$$\boxed{\deg(\partial) := \sup\{\mathfrak{o}(C) + 1 : C \textit{ occurs as cut formula in } \partial\}}.$$

In the crucial case

$$\boxed{(\partial : \Gamma \cup \Pi) = \frac{(\partial_1 : C, \Gamma) \qquad (\partial_2 : \overline{C}, \Pi)}{\Gamma \cup \Pi} \;(\text{Cut})}$$

we let $(\mathcal{E}(\partial) : \Gamma \cup \Pi) :=$

$$\boxed{\left(\mathcal{E}\left(\mathcal{R}\left(\frac{(\mathcal{E}(\partial_1) : C, \Gamma) \quad (\mathcal{E}(\partial_2) : \overline{C}, \Pi)}{\Gamma \cup \Pi} \;(\text{Cut})\right)\right) : \Gamma \cup \Pi\right)},$$

where corresponding *cut reduction operation* $\partial \hookrightarrow \mathcal{R}(\partial)$ satisfies

$$\boxed{\deg(\mathcal{R}(\partial)) < \deg(\partial), \text{ if } \deg(\partial_1) = \deg(\partial_2) = 0}.$$

The $\mathcal{R}(\partial)$'s are obtained by standard predicative approach using symmetric form of basic inferences, propositional inversions, inversions of inferences $\langle\cup\rangle$, $[\cup]$, $\langle;\rangle$, $[;]$, $[*]$.
Moreover

$$\boxed{h(\mathcal{R}(\partial)) \leq \omega^{h(\partial_1)} \# \omega^{h(\partial_2)} < \omega^{h(\partial)}}$$

Star inferences $\langle*\rangle$ and $[*]$ are treated analogously to predicate rules $(\exists x)$ and $(\forall x)$, respectively.

§5. $\mathrm{SEQ}_\omega^{\mathsf{PDL}}$: Predicative cut elimination: Corollary

- Here and below we argue in **PA**$+\mathrm{TI}\,(<\varphi_\omega(0))$.

Corollary

Let Γ be any sequent that does not contain occurrences $[P^*]$. Suppose that Γ is derivable in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$. Then Γ is derivable in a subsystem of $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$ that does not contain inferences $[*]$ and/or (CUT) *(abbr.: $\mathrm{SEQ}_1^{\mathrm{PDL}}$).*
Note that every derivation in $\mathrm{SEQ}_1^{\mathrm{PDL}}$ is finite.

Proof.

Obvious by the subformula property of cutfree derivations. □

- Let $\mathcal{L}_0$ be the star-free sublanguage of $\mathcal{L}$. Denote by $\mathrm{SEQ}_0^{\mathrm{PDL}}$ the (finite) $\mathcal{L}_0$-subsystem of $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ that does not contain inferences $\langle * \rangle$, $[*]$, (CUT).

Theorem

Let $\Sigma = \langle P_1^* \rangle A_1, \cdots, \langle P_n^* \rangle A_n, \Pi$ $(n > 0)$ with $A_i, \Pi \in \mathcal{L}_0$. For any chosen n-tuple of natural numbers $\overrightarrow{k} = k_1, \cdots, k_n$ let $\widehat{\Sigma}_{\overrightarrow{k}}$ arise from Σ by replacing every $\langle P_i^* \rangle A_i$ by $A_i, \langle P_i \rangle A_i, \cdots, \langle P_i \rangle^{k_i} A_i$ $(1 \leq i \leq n)$. Then Σ is derivable in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ iff $\widehat{\Sigma}_{\overrightarrow{k}}$ is derivable in $\mathrm{SEQ}_0^{\mathrm{PDL}}$, for some $\overrightarrow{k}$.

Proof.

Follows by standard arguments from the Corollary. □

§7. $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$: PSPACE refinement

Let $\mathcal{L}_{01}$ be a sublanguage of $\mathcal{L}_0$ containing only atomic programs p and $\mathcal{L}_{00}$ be its p-free fraction. Let $\mathrm{SEQ}_{01}^{\mathrm{PDL}}$ be as follows :

$$(\mathrm{AX}) \quad x, \neg x, \Gamma$$

$$(\vee) \quad \frac{A, B, \Gamma}{A \vee B, \Gamma} \qquad (\wedge) \quad \frac{A, \Gamma \qquad B, \Gamma}{A \wedge B, \Gamma}$$

$$(\mathrm{GEN}) \quad \frac{A_1, \cdots, A_n}{(p)_{\chi_1} A_1, \cdots, (p)_{\chi_n} A_n, \Gamma} \quad (n > 0)$$
$$\text{if } \sum_{i=1}^{n} \chi_i = 1.$$

Lemma (p-separation)

Suppose that $[p]A_1, \cdots, [p]A_j, \langle p \rangle B_1, \cdots, \langle p \rangle B_k, \Gamma$, *where* $\Gamma = (q_1)C_1, \cdots, (q_l)C_l, L_1, \cdots, L_n$ *for literals* L_i *and* $p \neq q_j$, *is derivable in* $\mathrm{SEQ}_{01}^{\mathrm{PDL}}$. *Then so is either* Γ *or* $A_i, B_1, \cdots, B_k$, *for some* $i \in [1, j]$, *without increasing the height of derivation.*

Theorem

The derivability in $\mathrm{S}\mathrm{EQ}_{01}^{\mathrm{PDL}}$ *is a PSPACE problem.*

Proof.

By the p-separation lemma we can turn (at most binary) natural proof search tree ∂ with root sequent Σ into a Boolean circuit C_∂ with binary AND, OR and unary ID gates, where $\mathrm{ID}(x) := x$ for $x \in \{0,1\}$, such that AND, OR and ID correspond to inferences $(\wedge)$, $(\mathrm{G}\mathrm{EN})$ for $\sum_{i=1}^{n} \xi_i > 1$ and $(\vee)$ with $(\mathrm{G}\mathrm{EN})$ for $\sum_{i=1}^{n} \xi_i = 1$, respectively. By the subformula property, the depth of C_∂ is proportional to $|\Sigma|$. The truth evaluations are defined for C_∂ via $val\,(\Delta) := 1$ (**true**) iff $\Delta = (\mathrm{A}\mathrm{X})$, for every leaf Δ. Obviously $val\,(\Sigma) = 1$ iff ∂ is a correct derivation in $\mathrm{S}\mathrm{EQ}_{01}^{\mathrm{PDL}}$. By familiar techniques the whole evaluation is computable in $\mathcal{O}(|\Sigma|)$ space. □

§8. $\mathrm{SEQ}^{\mathrm{PDL}}_{\omega}$: Herbrand expansion: Special cases

- Recall that for any $\Sigma = \langle p^* \rangle A, \Pi$ with $A \in \mathcal{L}_{01}, \Pi \in \mathcal{L}_{00}$ the following holds. Suppose that Σ is derivable in $\mathrm{SEQ}^{\mathrm{PDL}}_{\omega}$. Then there exists a $k \geq 0$ such that $\widehat{\Sigma}_k := A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \Pi$ is derivable in $\mathrm{SEQ}^{\mathrm{PDL}}_{01}$. How large is k ?

Definition

Call *basic CNF*(p) (: *BCNF*(p)) any $A = \bigwedge_{i=1}^{m} \left(C_i \vee \bigvee_{j=1}^{k} (p)_{\chi_{ij}} D_{ij} \right)$, where $\chi_{ij} \in \{0,1\}$ and $C_i, D_{ij} \in \mathcal{L}_{00} \cup \{\emptyset\}$.

Theorem

Let $\widehat{\Sigma}_k = A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \Pi$ *for* $A = \bigwedge_{i=1}^{m} \left(C_i \vee \bigvee_{j=1}^{k} (p)_{\chi_{ij}} D_{ij} \right) \in \mathrm{BCNF}(p), \Pi \in \mathcal{L}_{00}$. *If* $\widehat{\Sigma}_k$ *is derivable in* $\mathrm{SEQ}^{\mathrm{PDL}}_{01}$ *then so is* $\widehat{\Sigma}_{\mathcal{O}(km^2)}$.

§8. $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$: Herbrand expansion: Special cases

Definition (dual form)

Let p be fixed. Call *basic DNF*(p) (abbr.: *BDNF*(p)) any formula $\bigvee_{i=1}^{m} \left(C_i \wedge (p)_{\chi_i} D_i \right) \in \mathcal{L}_{01}$ where $\chi_i \in \{0,1\}$ and $C_i, D_i \in \mathcal{L}_{00} \cup \{\emptyset\}$.

Problem

Let $\Sigma = \langle p^* \rangle A, B$ for $A \in \mathrm{BDNF}(p)$, $B \in \mathcal{L}_{00}$ and suppose that Σ is (not) derivable in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$.
Is $(\exists k)\mathrm{SEQ}_{0}^{\mathrm{PDL}} \vdash \widehat{A}_k \vee B$ (resp. $(\forall k)\mathrm{SEQ}_{0}^{\mathrm{PDL}} \nvdash \widehat{A}_k \vee B$) verifiable by TM in $|\Sigma|$-polynomial space, where $\widehat{A}_k = A \vee \langle p \rangle A \vee \cdots \vee \langle p \rangle^k A$?

Bonus: If so, then **PSPACE** $=$ **EXPTIME** , as such Σ is known to be EXPTIME complete (: take e.g. negation of formula $\mathrm{ACCEPTS}_{M,x}$ stating that any satisfying Kripke frame encodes an accepting computation of a given polynomial-space alternating Turing machine M on a given input x over M's alphabet).