From Hall algebras to legendrian skein algebras

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Outline

(1) Local theory

- Representation theory of $GL(n, \mathbb{F}_q)$ and $\mathrm{Core}(D^b(\mathbb{F}_q))$
- Braids and legendrian tangles

(2) Global theory

- Fukaya categories of surfaces and their Hall algebras
- Legendrian skein algebras

Based on `arXiv:1908.10358`, `arXiv:1910.04182`, and ongoing joint work with Ben Cooper.

Representation theory of $GL(n, \mathbb{F}_q)$

"Philosophy of cusp forms", case $G_n := GL(n, \mathbb{F}_q)$

(1) Cuspidal representations of $GL(n, \mathbb{F}_q)$ correspond to characters

$$\mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$$

not factoring through $\mathbb{F}_{q^{n-1}}^\times$.

(2) From cuspidals, get everything else by parabolic induction: partition $n = n_1 + \ldots + n_k$, V_i representation of G_{n_i}, then pull-push along the span

$$G_{n_1} \times \cdots \times G_{n_k} \longleftarrow \{\text{block upper-triangular matrices}\} \longrightarrow G_n$$

is representation $V_1 \circ \cdots \circ V_k$ of G_n.

Unipotent representations

Take trivial representation $\mathbb{C}$ of $GL(1, \mathbb{F}_q)$ . . . simplest cuspidal representation

Parabolic induction gives

$$\mathbb{C} \circ \cdots \circ \mathbb{C} = \mathbb{C}^{G_n/B}$$

where

- $B \subset G_n$ subgroup of upper triangular matrices
- $G_n/B =$ complete flags in $\mathbb{F}_q^n$
- $\mathbb{C}^{G_n/B} =$ functions $G_n/B \to \mathbb{C}$

Taking summands & direct sums $\longrightarrow$ unipotent representations

Iwahori–Hecke algebra of type A_{n-1}: Generators

Endomorphisms of representation $\mathbb{C}^{G_n/B}$:

$$\mathrm{End}_{G_n}(\mathbb{C}^{G_n/B}) = \mathbb{C}^{B\backslash G_n/B}$$

Bruhat decomposition: $B\backslash G_n/B \cong S_n$

Transposition $(i, i+1) \in S_n \leftrightarrow$ operator T_i on $\mathbb{C}^{G/B}$ mapping flag

$$0 = E_0 \subset E_1 \subset \ldots \subset E_i \subset \ldots \subset E_n = \mathbb{F}_q^n$$

to sum of q flags

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{i-1} \subset E_i' \subset E_{i+1} \subset \ldots \subset E_n = \mathbb{F}_q^n$$

with $E_i' \neq E_i$.

Iwahori–Hecke algebra of type A_{n-1}: Relations

Complete set of relations among T_i:

- Skein relation:

$$T_i^2 = (q-1)T_i + q, \qquad 1 \le i \le n-1$$

- Braid relations:

$$\begin{aligned} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad 1 \le i \le n-2 \\ T_iT_j = T_jT_i, \qquad 1 \le i,j \le n-1, |i-j|>1 \end{aligned}$$

Relations polynomial in $q \implies \exists$ generic Iwahori–Hecke algebra over $\mathbb{C}[q]$

Specialization $q=1$ gives group algebra $\mathbb{C}[S_n]$

Categorical reformulation

Embedding of monoidal category of braids/skein relations:

- **Objects**: finite subsets of $\mathbb{R}$ modulo isotopy = $\mathbb{Z}_{\geq 0}$
- **Morphisms** $n \to n$: $\mathbb{C}$-linear combinations of braids of n strands modulo isotopy & skein relation
- **Composition**: concatenation of braids
- **Monoidal product**: stacking of braids

into category of functors

$$\mathrm{Core}\left(\mathrm{Vect}_{\mathbb{F}_q}^{\mathrm{fd}}\right) \longrightarrow \mathrm{Vect}_{\mathbb{C}}^{\mathrm{fd}}$$

from underlying groupoid of $\mathrm{Vect}_{\mathbb{F}_q}^{\mathrm{fd}}$,
monoidal product = parabolic induction

Categorical reformulation — remarks

Functor from braids/skein relations to representations of $\mathrm{Core}\left(\mathrm{Vect}_{\mathbb{F}_q}^{\mathrm{fd}}\right)$

- Target category is semisimple (representations of finite groups)
- Source category is $\mathbb{C}$-linear, but does not have sums & summands
- Closure of embedded image in target category is category of unipotent representations
- Irreducible unipotent representations indexed by partitions (c.f. irreducible representations of symmetric group)

Extension to complexes

Replace $\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_q}$ by its bounded derived category

$$\mathcal{D} := D^b\left(\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_q}\right)$$

and consider category of functors

$$\mathrm{Core}(\mathcal{D}) \longrightarrow \mathrm{Vect}^{\mathrm{fd}}_{\mathbb{C}}$$

Monoidal product is pull-push along span of ∞-groupoids (homotopy types):

$$\mathrm{Core}(\mathcal{D}) \times \mathrm{Core}(\mathcal{D}) \longleftarrow \mathrm{Core}\left(\mathrm{Fun}(\bullet \to \bullet, \mathcal{D})\right) \longrightarrow \mathrm{Core}(\mathcal{D})$$

$$(A, C) \qquad \longleftarrow \qquad A \to B \to C \to A[1] \qquad \longrightarrow \qquad B$$

Complexes and legendrian tangles

For representations of $\mathrm{Core}(D^b(\mathbb{F}_q))$, turns out we need *legendrian* tangles!

$\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_q}$	braids
$D^b(\mathbb{F}_q)$	graded legendrian tangles

Local picture of legendrian curves

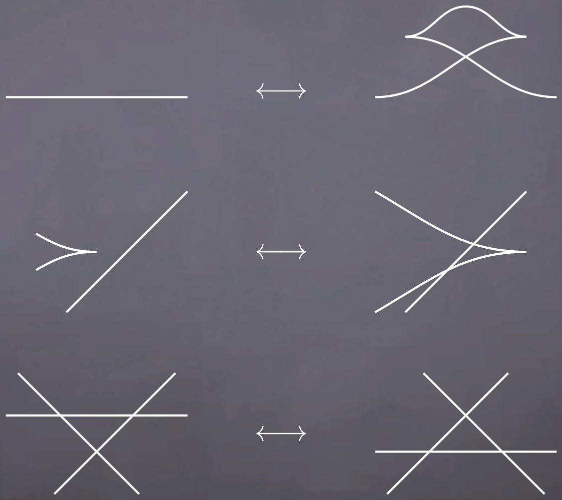
Legendrian curve: 1-form $dz - ydx$ vanishes along tangent direction

Under xz-projection (front) $y = dz/dx \implies$

- downward branch over upward branch at crossing

- slope never vertical
- front of generic legendrian curve can have left & right cusps

Legendrian Reidemeister moves (front projection)



Grading of legendrian curves

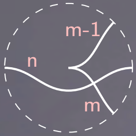
Assignment of integer to each strand ending at cusps

Condition at cusp: increase by 1 on lower strand

Equivalently: choice of $\mathrm{Arg}(dx + idy)$ along curve ($\implies$ image in xy-plane should have total winding number 0)

Generalizes to contact 3-fold M with given rank 1 subbundle of contact bundle $\subset TM$

Legendrian skein relations (front projection)



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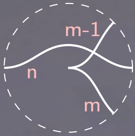
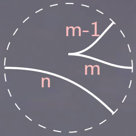
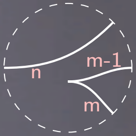
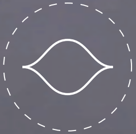


$= \delta_{m,n} z$


$- \delta_{m,n+1} z$



$= z^{-1}$



$= 0$

$z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, $\delta_{m,n}$ = Kronecker delta

Category of graded legendrian tangles

- **Objects:** finite $\mathbb{Z}$-graded subsets X of $\mathbb{R}$ up to isotopy (grading = function $\deg : X \to \mathbb{Z}$)

- **Morphisms:** $\mathrm{Hom}(X, Y) =$ vector space $/\mathbb{C}$ generated by isotopy classes of tangles L with left boundary $\partial_0 L = Y$ and right boundary $\partial_1 L = X$ modulo the skein relations ($q =$ prime power).

- **Composition:** horizontal composition (concatenation) of tangles

- **Monoidal product:** vertical composition (stacking) of tangles

Mapping graded subsets of $\mathbb{R}$ to representations

Notation: $\mathbb{C}_G$ = trivial 1-dim representation of G

Mapping a singleton:

$$\overset{n}{\bullet} \quad \mapsto \quad \mathbb{C}_{\mathrm{Aut}(\mathbb{F}_q[-n])}$$

For larger graded $X \subset \mathbb{R}$ determined by compatibility with $\otimes$:

$$X \quad \mapsto \quad \bigoplus_{\delta} \mathbb{C}_{\mathrm{Aut}(H^\bullet(\mathbb{F}_q X, \delta))}$$

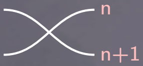
where sum is over combinatorial differentials: injective maps

$$X \supset \mathrm{Dom}(\delta) \xrightarrow{\delta} X \setminus \mathrm{Dom}(\delta)$$

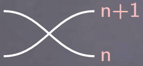
of degree 1, decreasing with respect to order induced from $\mathbb{R}$

Mapping graded legendrian tangles to intertwiners

$\mapsto$ $q^{-\frac{1}{2}}T : \mathbb{C}^{\mathbb{P}^1(\mathbb{F}_q)} \to \mathbb{C}^{\mathbb{P}^1(\mathbb{F}_q)}$

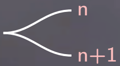


$\mapsto$ projection to $\mathbb{C}_{\mathrm{Aut}(\mathbb{F}_q[-n]\oplus\mathbb{F}_q[-n-1])}$

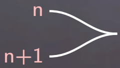


$\mapsto$ inclusion of $\mathbb{C}_{\mathrm{Aut}(\mathbb{F}_q[-n]\oplus\mathbb{F}_q[-n-1])}$

$\mapsto$ identity on $\mathbb{C}_{\mathrm{Aut}(\mathbb{F}_q[-m]\oplus\mathbb{F}_q[-n])}$, $|m-n|>1$



$\mapsto$ $z^{-1}\cdot$ projection to $\mathbb{C}_{\mathrm{Aut}(0)}$



$\mapsto$ inclusion of $\mathbb{C}_{\mathrm{Aut}(0)}$

Main theorem of local theory

Theorem: The mapping defined above gives a well defined fully faithful functor from the category of graded legendrian tangles modulo skein relations to the category of representations of the underlying groupoid of $D^b(\mathbb{F}_q)$.

- This was proven, in a somewhat different formulation, in *Flags and Tangles* [arXiv:1910.04182].
- The functor extends the prototypical functor from braids (in degree 0) to representations of the underlying groupoid of $\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_q}$ discussed before, the same remarks apply.

From local to global

- Disk with two marked points on the boundary (implicitly the setting above) $\rightsquigarrow$ surface with marked points

- Goal: Show graded legendrian skein algebra appears as subalgebra of Hall algebra of Fukaya category

- Strategy: Glue (form coend) along categories considered in local theory

Hall correspondence

$\mathcal{C}$ — triangulated DG-category

$$\mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{C}) \longleftarrow \mathrm{Core}(\mathcal{C}^{\mathcal{A}_2}) \longrightarrow \mathrm{Core}(\mathcal{C})$$

$$(A, C) \longmapsfrom A \to B \to C \to \longmapsto B$$

Various versions of Hall algebra obtained by applying pull-push functors to this span of ∞-groupoids (point of view advocated by Dyckerhoff–Kapranov in *Higher Segal Spaces*)

Homotopy cardinality

π-finite space: $\pi_i(X)$ finite for $i \geq 0$ and vanishes for $i \gg 0$, has homotopy cardinality (Baez–Dolan):

$$|X|_h := \sum_{x \in \pi_0(X)} \prod_{i=1}^{\infty} |\pi_i(X, x)|^{(-1)^i}$$

Given map $\phi : X \to Y$ of π-finite spaces get

$$\mathbb{Q}\pi_0(X)_c \underset{\phi_*}{\overset{\phi_!}{\rightleftarrows}} \mathbb{Q}\pi_0(Y)_c$$

$$\phi^* f := f \circ \pi_0(\phi), \qquad (\phi_! f)(y) := \sum_{\substack{x \in \pi_0(X) \\ \phi(x) = y}} |\mathrm{hofib}(\phi|_x)|_h \, f(x)$$

where $\mathbb{Q}\pi_0(X)_c :=$ functions $f : \pi_0(X) \to \mathbb{Q}$ with finite support

Hall algebra of triangulated DG-category (Toen)

Apply homotopy cardinality formalism to Hall correspondence of triangulated DG-category $\mathcal{C}$ (satisfying finiteness conditions):

$\mathrm{Hall}(\mathcal{C})$ = finite $\mathbb{Q}$-linear combinations of isomorphism classes of objects of $\mathcal{C}$

Explicit formula for structure constants:

$$g_{A,C}^{B} = \frac{|\mathrm{Ext}^0(A,B)_C| \cdot \prod_{i=1}^{\infty} \left|\mathrm{Ext}^{-i}(A,B)\right|^{(-1)^i}}{|\mathrm{Aut}(A)| \cdot \prod_{i=1}^{\infty} \left|\mathrm{Ext}^{-i}(A,A)\right|^{(-1)^i}}$$

where $\mathrm{Ext}^0(A,B)_C :=$ morphisms $A \to B$ with cone C

Surfaces with Liouville and grading structure

(1) S — compact surface with boundary
(2) $N \subset \partial S$ — finite set of marked points
(3) θ — Liouville 1-form on S:

- $d\theta$ nowhere vanishing (area form)
- vector field Z with $i_Z d\theta = \theta$ points outwards along ∂S

(4) $\eta \in \Gamma(S, \mathbb{P}(TS))$ — grading structure on S (foliation)

From this data construct:

- Fukaya category $\mathcal{F}(S, N, \theta, \eta; \mathbb{F})$ — linear A_∞/DG-category over field $\mathbb{F}$, triangulated
- Contact 3-fold $S \times \mathbb{R}$ with contact form $dz + \theta$ and its (graded, legendrian) skein algebra

Fukaya category of a disk

$$\mathcal{F}(\text{disk with } n+1 \text{ marked points on boundary}) \cong \mathcal{A}_n$$

where

$$\mathcal{A}_n := D^b(\underbrace{\bullet \to \bullet \to \ldots \to \bullet}_{n \text{ vertices}})$$

is the bounded derived category of representations of A_n-type quiver over $\mathbb{F}$ (independent of orientation of arrows)

Equivalently, an object of $\mathcal{A}_n$ can be described as filtered acyclic complex

$$0 = F_0C \subset F_1C \subset \ldots \subset F_nC \subset F_{n+1}C = C \sim 0$$

and the i-th boundary functor $\mathcal{A}_n \to \mathcal{A}_1$ sends this to the chain complex $F_iC/F_{i-1}C$, $1 \leq i \leq n+1$.

Fukaya category of a surface — gluing

Surface glued to itself along pair of marked points on the boundary:

then $\mathcal{F}(S')$ can be computed (or defined inductively) as homotopy equalizer of DG-categories:

$$\mathcal{F}(S') \longrightarrow \mathcal{F}(S) \rightrightarrows \mathcal{A}_1$$

where pair of parallel arrows are boundary functors corresponding to pair of marked points

Fukaya category of a surface — example

Example: $\mathcal{F}(S)$ = annulus with marked point on each boundary component

$\mathcal{F}(S)$ computed as coequalizer of DG-categories:

$$\mathcal{F}(S) = D^b(\bullet \rightrightarrows \bullet) \longrightarrow \mathcal{A}_2 \oplus \mathcal{A}_2 \rightrightarrows \mathcal{A}_1 \oplus \mathcal{A}_1$$

Note that $\mathcal{F}(S) \cong D^b(\mathrm{Coh}(\mathbb{P}^1(\mathbb{F}_q)))$ — simple example of homological mirror symmetry

Skein algebra of $S \times \mathbb{R}$

- Generated by graded legendrian links in $S \times \mathbb{R}$, allowed to have endpoints in $N \times \mathbb{R}$
- Impose same skein relations as for tangles before, if $N \neq \emptyset$ also have boundary versions of the skein relation
- Algebra product given by stacking links on top of each other
- For our purposes, coefficient ring is $\mathbb{C}$ and q is a fixed prime power, but could also define with $q^{\frac{1}{2}}$ a formal variable

Skein algebra — gluing

- Skein algebra itself does to satisfy same gluing axiom as Fukaya category, need variant with frozen boundary condition at subset of $N \subset \partial S$: boundary of link is fixed graded subset $X \subset \mathbb{R}$

- Varying X gives lax monoidal functor from category of graded Legendrian tangles, $\mathcal{S}$, to $\mathrm{Vect}_{\mathbb{C}}^{fd}$ (i.e. $\mathcal{S}$-module)

- For boundary condition at several points in N, get functor from $\otimes$-product of copies of $\mathcal{S}$

- Gluing pair of boundary marked points corresponds to taking coend of bifunctor ($\otimes$-product of $\mathcal{S}$-module with itself)

Hall algebra — gluing

- As for skein algebra, need to use variant of Hall algebra with boundary condition: framing (i.e. isomorphism with fixed object X) of image under boundary functor

- Varying X gives lax monoidal functor from category of representations of $\mathrm{Core}(D^b(\mathbb{F}_q))$, to $\mathrm{Vect}_{\mathbb{C}}^{fd}$ (i.e. $\mathcal{S}$-module)

- Gluing (equalizer) corresponds to taking coend

- Semisimplicity of category of representations makes coend very computable!

Main theorem

- (S, N) — marked Surface with Liouville form θ and grading η as before
- $\mathbb{F}_q$ — finite field

Theorem: There is an injective homomorphism of associative algebras

$$\mathrm{Skein}(S, N, \eta, \theta, q) \hookrightarrow \mathrm{Hall}(\mathcal{F}(S, N, \eta, \theta, \mathbb{F}_q))$$

from the legendrian skein algebra to the Hall algebra of the Fukaya category.

The homomorphism was already constructed in *Legendrian skein algebras and Hall algebras* [arXiv:1908.10358], the injectivity part is work in progress jointly with Ben Cooper

Open problems and further directions

(1) $\mathbb{Z}/n$ grading — issue with homotopy cardinality

(2) More sophisticated variants of Hall algebra: motivic/cohomological

(3) $q = 1$ limit, categories "over $\mathbb{F}_1$"?

(4) categorification of skein algebra

(5) higher dimensional contact manifolds (simplest case: $J^1 M = T^* M \times \mathbb{R}$)

— end —