CHOW-WITT GROUPS, RAMIFICATION AND QUADRATIC FORMS
HIM WORKSHOP ON PERIODS
JAN. 14-19, 2018

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Introduction

Using the Milnor-Witt K-sheaves as a "quadratic refinement' of the Milnor K-sheaves, one can refine a number of contructions, such as intersection theory or top Chern classes of vector bundles, from the Chow ring to the *Chow-Witt ring* (constructed by Barge-Morel [2] and studied in detail by Fasel [6] and others). In some cases, this allows one to refine the classical degree of an intersection or characteristic class to a degree with values in the Grothendieck-Witt ring $\mathrm{GW}(k)$ of the base-field k. Examples of such refinements include the topological Euler characteristic as well as the Riemann-Hurwicz formula. There is also a conjectural relation with periods.

1. Milnor-Witt sheaves

For simplicity we work over a perfect base-field k with $\mathrm{char} k \neq 2$. We write $\mathbf{Sm}/k$ for the category of smooth separated finite-type k-schemes.

For a field F, the Milnor K-theory of F is the graded ring $K_*^M(F)$ defined by generators and relations:

• Generators: for $u \in F^\times$ one has the generator $\{u\}$ in degree $+1$.
• Relations:
(1) (Additivity) $\{uv\} = \{u\} + \{v\}$ for $u, v \in F^\times$
(2) (Steinberg relation) $\{u\}\{1-u\} = 0$ for $u \in F \setminus \{0, 1\}$

For a DVR $\mathcal{O}$ with quotient field K and residue field k, there is a map $\partial : K_n^M(K) \to K_{n-1}^M(k)$, which allows one to define the unramified sheaf $\mathcal{K}_n^M$ on $\mathbf{Sm}/k$ (for the Nisnevich topology). $\mathcal{K}_0^M$ is the constant sheaf $\mathbb{Z}$, $\mathcal{K}_1^M = \mathbb{G}_m$.

We have the Bloch/Kato formula relating the Chow ring with the cohomology of the Milnor sheaves:

$$\mathrm{CH}^n(X) \cong H^n(X, \mathcal{K}_n^M).$$

For $f : Y \to X$ a morphism in $\mathbf{Sm}/k$ we have the pull-back map $f^* : H^p(X, \mathcal{K}_q^M) \to H^p(Y, \mathcal{K}_q^M)$, which gives rise to the classical pull-back map $f^* : CH^n(X) \to \mathrm{CH}^n(Y)$. If $f : Y \to X$ is proper of relative dimension d, there is a push-forward $f_*: H^p(Y, \mathcal{K}^M_q) \to H^{p-d}(X, \mathcal{K}^M_{q-d})$ which gives rise to the push-foward $f_*: \mathrm{CH}^n(Y) \to \mathrm{CH}^{n-d}(X)$.

Hopkins-Morel (see e.g. [11]) have defined the Milnor-Witt sheaf $\mathcal{K}^{MW}_*$ as a "quadratic refinement" of the Milnor sheaf $\mathcal{K}^M_*$. For a field F, $K^{MW}_*(F)$ is the graded ring defined by generators and relations:

• Generators: for $u \in F^\times$ one has the generator $[u]$ in degree +1.There is an additional generator η in degree -1.
• Relations:
(0) $\eta[u] = [u]\eta$ for all $u \in F^\times$
(1) (Twisted additivity) $[uv] = [u] + [v] + \eta[u][v]$ for $u, v \in F^\times$.
(2) (Steinberg relation) $[u][1-u] = 0$ for $u \in F \setminus \{0, 1\}$
(3) (Hyperbolic relation) Let $h = 2 + \eta[-1]$. Then $\eta \cdot h = 0$.

For a DVR $\mathcal{O}$ with quotient field K, residue field k and generator t for the maximal ideal, there is a map $\partial_t : K^{MW}_n(K) \to K^{MW}_{n-1}(k)$. Although ∂_t depends on t, this still gives rise to the unramified (Nisnevich) sheaf $\mathcal{K}^{MW}_n$ on $\mathbf{Sm}/k$.

Let $\mathrm{GW}(F)$ denote the Grothendieck-Witt ring of non-degenerate quadratic forms over F: this is the group completion of the monoid (under orthogonal direct sum) of non-degenerate quadratic forms over F. The hyperbolic form is the rank 2 form $H(x, y) = x^2 - y^2$, and the Witt ring $W(F)$ is the quotient $\mathrm{GW}(F)/(H)$. Note that $(H) = \mathbb{Z} \cdot H$, as for a quadratic form q of rank n, one has $q \cdot H = n \cdot H$. For $u \in F^\times$ we let q_u be the rank one form $q_u(x) = ux^2$.

For $u \in F^\times$ let $\langle u \rangle = 1 + \eta[u] \in K^{MW}_0(F)$.

Theorem 1.1 (Morel [11,]). *Sending* $q_u \in \mathrm{GW}(F)$ *to* $\langle u \rangle \in K^{MW}_0(F)$ *extends to an isomorphism of rings* $\mathrm{GW}(F) \to K^{MW}_0(F)$. *Sending* $q_u \in \mathrm{GW}(F)$ *to* $\eta^n\langle u \rangle \in K^{MW}_{-n}(F)$ *descends to an isomorphism of* $\mathrm{GW}(F)$-*modules* $W(F) \to K^{MW}_{-n}(F)$ *for all* $n \geq 1$.

Sending a unit u to the element $\langle u \rangle$ gives a map of sheaves of abelian groups $\mathbb{G}_m \to \mathcal{K}^{MW\times}_0$, which via the multiplication $\mathcal{K}^{MW}_0 \times \mathcal{K}^{MW}_* \to \mathcal{K}^{MW}_*$ defines an action of $\mathbb{G}_m$ on $\mathcal{K}^{MW}_*$. For a line bundle $L \to X$ on some $X \in \mathbf{Sm}/k$, we have the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$ of nowhere zero sections of L and we define the twisted Milnor-Witt sheaf as $\mathcal{K}^{MW}_*(L) := \mathcal{K}^{MW}_* \times_{\mathbb{G}_m} \mathcal{L}^\times$. Note that there is a canonical isomorphism $\mathcal{K}^{MW}_*(L \otimes M^{\otimes 2}) \cong \mathcal{K}^{MW}_*(L)$, since $\langle u \rangle = \langle uv^2 \rangle$. Morel's isomorphism $K^{MW}_0(F) \cong \mathrm{GW}(F)$ extends to an identification of $\mathcal{K}^{MW}_0(L)$ with $\mathcal{GW}(L)$, the sheaf of L-valued quadratic forms (group completed as for GW).

Sending η to zero defines a surjection $\pi : \mathcal{K}^{MW}_n(L) \to \mathcal{K}^M_n$ (of sheaves on X). As multiplication by h kills η, this induces the *hyperbolic map* $h : \mathcal{K}^M_n \to \mathcal{K}^{MW}_n(L)$ and $\pi \circ h$ is multiplication by 2. For $n = 0$, the map $\pi : \mathcal{K}^{MW}_0(L) \to \mathcal{K}^M_0$ is the sheafified rank homomorphism.

Definition 1.2. Let $L \to X$ be a line bundle on some $X \in \mathbf{Sm}/k$. Define the twisted Chow-Witt group $\tilde{\mathrm{CH}}^n(X;L) := H^n(X, \mathcal{K}_n^{MW}(L)$.

The original definition of Barge-Morel is somewhat different, relying on combining Milnor K-theory and the powers of the augementation ideal in GW, but subsequent developments show the two definitions yield the same groups.

For $f : Y \to X$ and $L \to X$ a line bundle, we have $f^* : H^p(X, \mathcal{K}_q^{MW}(L)) \to H^p(Y, \mathcal{K}_q^{MW}(f^*L))$, inducing $f^* : \tilde{\mathrm{CH}}^n(X;L) \to \tilde{\mathrm{CH}}^n(Y;f^*L)$. For $f : Y \to X$ proper of relative dimension d, we have $f_* : H^p(Y, \mathcal{K}_q^{MW}(\omega_{Y/k} \otimes f^*L)) \to H^{p-d}(X, \mathcal{K}_{q-d}^{MW}(\omega_{X/k} \otimes L))$, inducing $f_* : \tilde{\mathrm{CH}}^n(Y;\omega_{Y/k} \otimes f^*L) \to \tilde{\mathrm{CH}}^{n-d}(X, \omega_X \otimes L)$. See [2, 6] for details.

For $\pi_X : X \to \operatorname{Spec} k$ a smooth and proper k-scheme of dimension n, we have the degree map

$$\deg_k := \pi_{X*} : \mathrm{CH}^n(X) \to \mathrm{CH}^0(\operatorname{Spec} k) = \mathbb{Z}$$

and its quadratic refinement

$$\tilde{\deg}_k := \pi_{X*} : \tilde{\mathrm{CH}}^n(X;\omega_{X/k}) \to \tilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k).$$

2. Euler classes and Euler characteristic

Definition 2.1. Let $V \to X$ be a rank r vector bundle on $X \in \mathbf{Sm}/k$ with zero-section $s_0 : X \to V$. The *Euler class* $e(V) \in \tilde{\mathrm{CH}}^r(X; \det^{-1} V)$ is defined as

$$e(V) := s_0^* s_{0*}(1_X)$$

where $1_X \in \tilde{\mathrm{CH}}^0(X) = H^0(X, \mathcal{GW})$ is the unit section.

For X smooth and projective over $\operatorname{Spec} k$, the *quadratic Euler characteristic* $\chi(X/k) \in \mathrm{GW}(k)$ is defined as

$$\chi(X/k) := \tilde{\deg}_k(e(T_{X/k}))$$

where $T_{X/k} = \Omega_{X/k}^\vee$ is the tangent bundle.

Note that $e(T_{X/k})$ lives in $\tilde{\mathrm{CH}}^{\dim_k X}(X;\omega_{X/k})$, so its quadratic degree is defined.

Here are some properties of the Euler class and Euler characteristic.

1. For $X \in \mathbf{Sm}/k$ projective over k, $\mathrm{rnk}(\chi(X/k)) \in \mathbb{Z}$ is the "topological" Euler characteristic of X, which one can for example define as

$$\chi^{top}(X) := \sum_{i=0}^{2\dim_k X} (-1)^i \dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell),$$

where $\bar{X} := X \times_k \bar{k}$, $\bar{k}$ an algebraic closure of k, and ℓ is a prime different from char$k$. If k is a subfield of $\mathbb{C}$, one can also use singular cohomology of $X(\mathbb{C})$ to define $\chi^{top}(X)$, that is $\chi^{top}(X)$ is the usual topological Euler

characteristic of $X(\mathbb{C})$.
2. If we have a real embedding $\sigma : k \hookrightarrow \mathbb{R}$, σ induces the signature map

$$\mathrm{sig}^\sigma : \mathrm{GW}(k) \to \mathbb{Z}$$

by composing $\sigma_* : \mathrm{GW}(k) \to \mathrm{GW}(\mathbb{R})$ with the signature map $\mathrm{sig} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$. Then

$$\mathrm{sig}^\sigma(\chi(X/k)) = \chi^{top}(X^\sigma(\mathbb{R})).$$

3. For $X \in \mathbf{Sm}/k$ projective over k, the T-suspension spectrum $\Sigma^\infty_T X_+$ in the motivic stable homotopy category $\mathrm{SH}(k)$ (a symmetric monoidal category) is strongly dualizable [12, §2], hence there a well-defined trace

$$\mathrm{Tr}(\mathrm{Id}_{\Sigma^\infty_T X_+}) \in \mathrm{End}_{\mathrm{SH}(k)}(1_{\mathrm{SH}(k)}).$$

By a theorem of Morel, there is a canonical identification

$$\mathrm{End}_{\mathrm{SH}(k)}(1_{\mathrm{SH}(k)}) \cong \mathrm{GW}(k)$$

and the corresponding element $\mathrm{Tr}(\mathrm{Id}_{\Sigma^\infty_T X_+}) \in \mathrm{GW}(k)$ is the Euler characteristic $\chi(X/k)$. One can thus extend the definition of $\chi(X/k)$ to not necessarily proper $X \in \mathbf{Sm}/k$ by setting $\chi(X/k) = \mathrm{Tr}(\mathrm{Id}_{\Sigma^\infty_T X_+}) \in \mathrm{GW}(k)$, assuming that $\Sigma^\infty_T X_+$ is strongly dualizable. This is the case for all $X \in \mathbf{Sm}/k$ if k admits resolution of singularities, or in general if X admits a smooth compactification $\bar{X} \supset X$ with $\bar{X} \setminus X$ a simple normal crossing divisor.

This interpretation gives us some additional properties of the Euler characteristic. A result of May [10, Theorem 0.1] says that in $\mathrm{SH}(k)$, $\mathrm{Tr}(\mathrm{Id})$ is additive in distinguished triangles of strongly dualizable objects. Assuming for instance resolution of singularities for k, this gives us:

i. For $X = U \cup V$, $U, V \subset X$ open subschemes, we have

$$\chi(X/k) = \chi(U/k) + \chi(V/k) - \chi(U \cap V/k).$$

ii. Let $\pi : E \to X$ be a Zariski locally trivial bundle with fiber F (and group $\mathrm{Aut}(F)$). Then $\chi(E/k) = \chi(F/k) \cdot \chi(X/k)$.
iii. $\chi(\mathbb{P}^n) = \sum_{i=0}^n \langle -1 \rangle^i$, so if $\mathbb{P} \to X$ is a Zariski locally trivial $\mathbb{P}^n$-bundle, then

$$\chi(\mathbb{P}/k) = \chi(X/k) \cdot \sum_{i=0}^n \langle -1 \rangle^i.$$

More generally, let $X \in \mathbf{Sm}/k$ be a projective *cellular* variety: X has a stratification by locally closed subsets C_i with $C_i \cong \mathbb{A}^{n_i}$, then $\mathrm{CH}^n(X)$ is a free abelian group of finite rank for each n. Let $r_{ev} = \sum_i \mathrm{rnk}\,\mathrm{CH}^{2i}(X)$, $r_{odd} = \sum_i \mathrm{rnk}\,\mathrm{CH}^{2i+1}(X)$, then

$$\chi(X/k) = \sum_{j=0}^{\dim_k X} (\mathrm{rnk}\,\mathrm{CH}^i(X)) \cdot \langle -1 \rangle^i = \sum_{j=1}^{n_{ev}} 1 \cdot x_j^2 + \sum_{j=1}^{n_{odd}} (-1) \cdot y_j^2.$$

iv. Let $X = \mathrm{Spec}\, F$, with $F \supset k$ a finite separable extension. Then $\chi(X/k)$ is the trace form $\mathrm{Tr}_{F/k}(\langle 1 \rangle)$.
v. Let $Z \subset X$ be a smooth codimension r closed subscheme, $\tilde{X} \to X$ the

blow-up of X along Z. Then $\chi(\tilde{X}/k) = \chi(X/k) + \chi(Z) \cdot \sum_{i=1}^{r-1} \langle -1 \rangle^i$.

4. For $V \to X$ a rank r vector bundle with dual bundle $V^\vee$, $e(V) = (-\langle -1 \rangle)^r e(V^\vee)$.

5. If $V \to X$ is an odd rank bundle, then $\eta \cdot e(V) = 0$ in $H^r(X, \mathcal{K}^{MW}_{r-1}(\det^{-1} V)$. In consequence, if X is smooth and projective of odd dimension over k, then $\chi(X/k) = N \cdot H$ for some integer N.

6. Let $V \to X$ be a rank r vector bundle and let $L \to X$ be a line bunde such that $L \cong M^{\otimes 2}$ for some line bundle M on X. Then

$$e(V \otimes L) = e(V) + \left(c_1(M) \cdot \sum_{i=1}^{r} c_{r-i}(V) \cdot c_1(L)^{i-1} \right) \cdot h \tag{2.1}$$

where $c_1(M), c_1(L)$ and the $c_{r-i}(V)$ are all in $\mathrm{CH}^*(X)$, $c_0(V) := 1$, and $(-) \cdot h$ is the hyperbolic map

$$(-) \cdot h : H^r(X, \mathcal{K}^M_r) \to H^r(X, \mathcal{K}^{MW}_r(\det^{-1} V)).$$

See [9] for details. In their paper [7] Kass and Wickelgren define a class in $\mathrm{GW}(k)$ associated to a vector bundle (with some restrictions) on a smooth projective variety, and use this to get an interesting quadratic invariant of the "count" of lines on a smooth cubic surface in $\mathbb{P}^3_k$. It turns out that their invariant is exactly the quadratic degree of the Euler class of a vector bundle V, assuming the necessary condition on $\det V$ so that the quadratic degree of $e(V)$ is defined.

3. Local Euler classes

Let $V \to X$ be a rank r vector bundle with zero section $0_V \subset V$. There is a *Thom class* $th(V) \in H^r_{0_V}(V, \mathcal{K}^{MW}_r(\det^{-1} V))$. Let $s : X \to V$ be a section and $Z \subset X$ a closed subset with $\mathrm{supp}\,(s) \subset Z$. We define the *Euler class with supports* $e_Z(V; s) \in H^r_Z(X, \mathcal{K}^{MW}_r(\det^{-1} V))$ as

$$e_Z(V; s) = s^*(th(V))$$

Suppose that $\mathrm{rank} V = \dim_k X = n$ and that $\mathrm{supp}\,(s) = \{x_1, \ldots, x_r\}$ is a finite set of closed points of X. As

$$H^n_{\{x_1,\ldots,x_r\}}(X, \mathcal{K}^{MW}_n(\det^{-1} V)) = \oplus_{i=1}^r H^n_{x_i}(X, \mathcal{K}^{MW}_n(\det^{-1} V)),$$

we can write

$$e_{\{x_1,\ldots,x_r\}}(V; s) = \sum_{i=1}^r e_{x_i}(V; s); \quad e_{x_i}(V; s) \in H^n_{x_i}(X, \mathcal{K}^{MW}_n(\det^{-1} V)).$$

Kass and WIckelgren [8] have developed an explicit formula for the local class $e_x(V; s)$, $x = x_i$, as follows. For a line bundle L on X, $n = \dim_k X$, there is a purity isomorphism

$$H^n_x(X, \mathcal{K}^{MW}_n(L)) \cong \mathrm{GW}(k(x); L \otimes \omega^{-1}_{X/k} \otimes k(x))$$

where $\mathrm{GW}(k(x); L\otimes\omega_{X/k}\otimes k(x))$ is the Grothendieck-Witt group of $L\otimes\omega_{X/k}\otimes k(x)$-valued quadratic forms.

Choose parameters $t_1,\ldots,t_n\in\mathcal{O}_{X,x}$ and a local basis $e_1,\ldots,e_n$ of sections for V near x. Then $s=\sum_{i=1}^n s_ie_i$ with $s_i\in\mathfrak{m}_x\subset\mathcal{O}_{X,x}$. Write $s_i=\sum_{j=1}^n a_{ij}t_j$, $a_{ij}\in\mathcal{O}_{X,x}$, let $Q(s)=\mathcal{O}_{X,x}/(s_1,\ldots,s_n)$ and let E be the image in $Q(s)$ of $\det(a_{ij})$. Then $Q(s)$ is a $k(x)$-algebra, (E) is the socle of $Q(s)$, $(E)\cong k(x)$. Let $\varphi: Q(s)\to k(x)$ be a $k(x)$-linear map with $\varphi(E)=1$, and let

$$q_\varphi: Q(s)\to \det{}^{-1} V\otimes\omega_{X/k}^{-1}\otimes k(x)$$

be the quadratic form

$$q_\varphi(y)=\varphi(y^2)\cdot(e_1\wedge\ldots\wedge e_n)^{-1}\cdot(dt_1\wedge\ldots\wedge dt_n)^{-1}.$$

Then under the purity isomorphism $e_x(V,s)$ maps to $q_\varphi\in\mathrm{GW}(k(x);\det^{-1}V\otimes\omega_{X/k}^{-1}\otimes k(x))$.

4. Riemann-Hurwitz formula

Theorem 4.1 ([9, Corollary 12.3]). *Let $X\in\mathbf{Sm}/k$ be projective of dimension n, $C\in\mathbf{Sm}/k$ a projective curve and $f: X\to C$ a separable map, giving us the non-zero section $df\in H^0(X,\Omega_{X/k}\otimes f^*\omega_{C/k})$. We suppose that there is a line bundle M on C with $\omega_{C/k}\cong M^{\otimes 2}$ and that df has only the isolated zeros $x_1,\ldots,x_r$. Then*

$$\chi(X/k)=(-\langle -1\rangle)^n\cdot\sum_{i=1}^r \mathrm{Tr}_{k(x_i)/k}e_{x_i}(\Omega_{X/k}\otimes f^*\omega_{C/k}^{-1},df)$$
$$-\frac{(-1)^n}{2}\deg_k\left(c_{n-1}(\Omega_{X/k})\cdot f^*c_1(\omega_{C/k})\right)\cdot H$$

Here we use a chosen isomorphism $\omega_{C/k}\cong M^{\otimes 2}$ to view $e_{x_i}(\Omega_{X/k}\otimes f^*\omega_{C/k}^{-1},df)\in\mathrm{GW}(k(x);f^*\omega_{C/k})$ as an element of $\mathrm{GW}(k(x))$, and $\mathrm{Tr}_{k(x_i)/k}$ is the trace map $\mathrm{Tr}_{k(x_i)/k}:\mathrm{GW}(k(x_i))\to\mathrm{GW}(k)$.

Bemerkung. If k is not perfect, the result still holds if one assumes that each residue field $k(x_i)$ is separable over k.

5. Some examples

Curves

Let $f: X\to\mathbb{P}^1_k$ be a separable map with X a smooth projective geometrically integral curve over k of genus g_X. Since X has odd dimension, $\chi(X/k)$ is a multiple of H and since $\mathrm{rnk}(\chi(X/k))=\chi^{top}(X)=2-2g_X$, we have

$$\chi(X/k)=(1-g_X)\cdot H.$$

Suppose that f is ramified at points $x_1,\ldots,x_r\in X$, and let $y_i=f(x_i)\in\mathbb{P}^1$. Without loss of generality we may assume that the y_i are all in $\mathbb{P}^1\setminus\{0\}=\operatorname{Spec}k[T]$. Let $g_i(T)\in k[T]$ be the monic irreducible equation for. $y_i\in$

Spec $k[T]$ and let $s_i \in \mathcal{O}_{\mathbb{P}^1,y_i}$ be defined by $s_i := g(T)/g_i'(T)$. Then s_i is a parameter in $\mathcal{O}_{\mathbb{P}^1,y_i}$ and is "normalized" in that $ds_i = dT$ in $\Omega_{\mathbb{P}^1/k,y_i}$. We use homogeneous coordinates T_0, T_1 for $\mathbb{P}^1$ with $T = T_1/T_0$ and fix the isomorphism $\omega_{\mathbb{P}^1/k} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2}$ as the one which sends the generating section T_0^2 over $\mathbb{A}^1$ to dT.

Chose a parameter $t_i \in \mathcal{O}_{X,x_i}$. Then $f^*(ds_i) = v_i t_i^{e_i-1} dt_i$ for an integer $e_i \geq 2$ and a unit $v_i \in \mathcal{O}_{X,x_i}$. Let $\bar{v}_i$ denote the image of v_i in $k(x_i)^\times$. The Kass-Wickelgren formula gives

$$e_{x_i}(\Omega_{X/k} \otimes f^*\omega_{C/k}^{-1}; df) = \langle \bar{v}_i \rangle \cdot \sum_{i=0}^{e_i-2} \langle -1 \rangle^i \in \mathrm{GW}(k(x_i))$$

where we use the isomorphism $\omega_{\mathbb{P}^1/k} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2}$ to pass from $\mathrm{GW}(k(x_i); f^*\omega_{\mathbb{P}^1/k}^{-1})$ to $\mathrm{GW}(k(x_i))$. Our Riemann-Hurwitz formula then gives

$$(g_X - 1) \cdot H = (-\langle -1 \rangle) \cdot \chi(X/k) = \sum_{i=1}^{r} \left(\mathrm{Tr}_{k(x_i)/k} \langle \bar{v}_i \rangle \cdot \sum_{i=0}^{e_i-2} \langle -1 \rangle^i \right) + \deg f \cdot H$$

Taking ranks (noting that $\mathrm{rnk} H = 2$) recovers the classical Riemann-Hurwitz formula.

We note that

$$\langle \bar{v}_i \rangle \cdot \sum_{i=0}^{e_i-2} \langle -1 \rangle^i = \begin{cases} \frac{e_i-1}{2} \cdot H & \text{for } e_i \text{ odd,} \\ \langle \bar{v}_i \rangle + \frac{e_i-2}{2} \cdot H & \text{for } e_i \text{ even,} \end{cases}$$

and thus the quadratic Riemann-Hurwitz formula yields a relation on the local leading terms of df:

$$\sum_{i, e_i \text{ even}} \mathrm{Tr}_{k(x_i)/k} \langle \bar{v}_i \rangle = m \cdot H$$

for some integer m.

In case $\mathrm{char} k = 0$, or more generally if all the ramification in $f : X \to \mathbb{P}^1_k$ is tame, we have $f^*(s_i) = u_i t_i^{e_i}$ for a unit $u_i \in \mathcal{O}_{X,x_i}^\times$ and $v_i = e_i u_i$. In addition to the classical Riemann-Hurwitz formula, this yields

$$\sum_{i, e_i \text{ even}} \mathrm{Tr}_{k(x_i)/k} \langle e_i \bar{u}_i \rangle = m \cdot H$$

for some integer m.

Diagonal hupersurfaces

Up to now in all our examples $\chi(X/k)$ is either hyperbolic, or hyperbolic plus $\langle 1 \rangle$. We can also compute $\chi(X/k)$ for $X \subset \mathbb{P}^{n+1}_k$ a degree m *diagonal hypersurface*, that is, a smooth hypersurface defined by degree m polynomial $\sum_{i=0}^{n+1} a_i T^m$, $\prod_i a_i \neq 0$, $(m, \mathrm{char} k) = 1$. As $\chi(X/h)$ is hyperbolic if n is odd, we restrict to the case of even n.

All smooth hypersurfaces of fixed degree m in $\mathbb{P}^{n+1}_k$ have the same topological Euler characteristic $\chi^{top}(X)$. We let

$$A_{n,m} = \begin{cases} \frac{1}{2}(\chi^{top}(X) - 1) & \text{for } m \text{ odd,} \\ \frac{1}{2}(\chi^{top}(X) - 2) & \text{for } m \text{ even.} \end{cases}$$

Theorem 5.1 ([9,]). *Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth degree m hypersurface with defining equation $\sum_{i=0}^{n+1} a_i T^m = 0$ and with n even. Then $A_{n,m}$ is an integer and*

$$\chi(X/k) = \begin{cases} A_{n,m} \cdot H + \langle m \rangle & \textit{for } m \textit{ odd,} \\ A_{n,m} \cdot H + \langle m \rangle + \langle -m \cdot \prod_{i=0}^{n+1} a_i \rangle & \textit{for } m \textit{ even.} \end{cases}$$

The idea of the proof is to take $Z \subset X$ to be the closed subscheme defined by $T_n = T_{n+1} = 0$ and let $\tilde{X}$ to be the blow-up of X along Z. Then $Z \subset \mathbb{P}^{n-1}_k$ is the diagonal hypersurface with defining equation $\sum_{i=0}^{n-1} a_i T^m = 0$, so we can use induction to compute $\chi(Z/k)$. The blow-up formula

$$\chi(\tilde{X}/k) = \chi(X/k) + \langle -1 \rangle \chi(Z/k)$$

tells us it suffices to compute $\chi(\tilde{X}/k)$. We have the morphism

$$f := (T_n : T_{n+1}) : \tilde{X} \to \mathbb{P}^1$$

which has as isolated critical points the closed subscheme of $\tilde{X}$ (or X) defined by $T_0 = \ldots = T_{n-1} = 0$, i.e. the closed subscheme x of $\mathbb{P}^{n+1}_k$ defined by $T_0 = \ldots = T_{n-1} = 0$, $a_n T_n^m + a_{n_1} T_{n+1}^m = 0$. One then uses the Kass-Wickelgren formula to compute the local term $e_x(\Omega_{\tilde{X}/k} \otimes f^* \omega^{-1}_{\mathbb{P}^1/k}, df)$ and then the Riemann-Hurwitz formula computes $\chi(\tilde{X}/k)$.

6. Some thoughts on the real realization

We write (X, x) for a k-scheme X pointed by an $x \in X(k)$. Set $\mathbb{G}_m = (\mathbb{A}^1 \setminus \{0\}, \{1\})$, $T = (\mathbb{P}^1, \infty)$, $X_+ = (X \amalg \operatorname{Spec} k, \operatorname{Spec} k)$. We let $\mathrm{SH}(k)$ denote the motivic stable homtopy category and SH the classical topological stable homotopy category. Objects of SH are spectra, that is, sequences of pointed spaces $(E_0, E_1, \ldots, E_n, \ldots)$, together with maps $\epsilon_n : \Sigma E_n \to E_{n+1}$, $\Sigma E_n := E_n \wedge S^1$. Objects of $\mathrm{SH}(k)$ are T-spectra ($T = (\mathbb{P}^1, \infty)$), that is, sequences $(E_0, E_1, \ldots, E_n, \ldots)$ with each E_n a point presheaf of spaces on $\mathbf{Sm}/k$, together with maps $\epsilon_n : \Sigma_T E_n \to E_{n+1}$, $\Sigma_T E_n := E_n \wedge T$. The suspension spectrum $\Sigma^\infty_T X_+ \in \mathrm{SH}(k)$ associated to an $X \in \mathbf{Sm}/k$ is the sequence $(X_+, X_+ \wedge T, \ldots, X_+ \wedge T^{\wedge n}, \ldots)$ with identity maps ϵ_n, just as the classical suspension spectrum $\Sigma^\infty X_+$ of a space X is the the sequence $(X_+, X_+ \wedge S^1, \ldots, X_+ \wedge S^n, \ldots)$ with identity maps ϵ_n ($S^n \cong (S^1)^{\wedge n}$).

Suppose we have an embedding $\sigma : k \to \mathbb{R}$. Sending $X \in \mathbf{Sm}/k$ to the space of $\mathbb{R}$-points $X(\mathbb{R})$ extends to a *real realization*

$$Re_\sigma : \mathrm{SH}(k) \to \mathrm{SH},$$

where SH is the classical stable homotopy category of spectra and $Re_\sigma(\Sigma^\infty_T X_+) = \Sigma^\infty X(\mathbb{R})_+$.

We recall that the unit in SH(k) is the motivic sphere spectrum

$$\mathbb{S}_k := \Sigma^\infty_T \mathrm{Spec}\, k_+ = (\mathrm{Spec}\, k_+, T, T^{\wedge 2}, \ldots).$$

The involution on $T \wedge T$ exchanging the two factors defined an involution $\tau : \mathbb{S}_k \to \mathbb{S}_k$, and as SH$(k)$ idempotently complete, we have a direct sum "eigenspace" decomposition in SH$(k)[1/2]$

$$\mathbb{S}_k = \mathbb{S}_k^+ \oplus \mathbb{S}_k^-$$

corresponding to writing $\mathrm{Id}_{\mathbb{S}_k} = (1/2)(\mathrm{Id}_{\mathbb{S}_k} - \tau) + (1/2)(\mathrm{Id}_{\mathbb{S}_k} + \tau)$. As $\mathbb{S}_k$ is the unit in SH(k), this gives the decomposition of SH$(k)_\mathbb{Q}$ as

$$\mathrm{SH}(k)_\mathbb{Q} = \mathrm{SH}(k)^+ \times \mathrm{SH}(k)^-$$

where τ acts by Id on SH$(k)^+$ and by $-$Id on sSH$(k)^-$.

We have already mentioned Morel's theorem, that $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \cong \mathrm{GW}(k) \cong K_0^{MW}(k)$; in fact Morel has computed (here $n, m \geq 0$):

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_T \mathbb{G}_m^{\wedge m}, \Sigma^\infty_T \mathbb{G}_m^{\wedge n+m}) \cong K_n^{MW}(k)$$

and for $n > 0$,

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_T \mathbb{G}_m^{\wedge n+m}, \Sigma^\infty_T \mathbb{G}_m^{\wedge m}) \cong K_{-n}^{MW}(k) \cong W(k).$$

In particular, there is an element in

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_T \mathbb{G}_m^{\wedge 2}, \Sigma^\infty_T \mathbb{G}_m) = \mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_T S^1 \wedge \mathbb{G}_m^{\wedge 2}, \Sigma^\infty_T S^1 \wedge \mathbb{G}_m)$$

corresponding to the degree -1 generator $\eta \in K_{-1}^{MW}(k)$.

This map has a very simple geometric description. In the unstable motivic homotopy category $\mathcal{H}_\bullet(k)$ there are canonical isomorphisms

$$T \cong (\mathbb{P}^1, 1) \cong S^1 \wedge \mathbb{G}_m,\ (\mathbb{A}^2 \setminus \{0\}, (1,1)) \cong S^1 \wedge \mathbb{G}_m^{\wedge 2}$$

and η is the stabilization of the map $\pi : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ sending $(x, y) \in \mathbb{A}^2 \setminus \{0\}$ to the line through (x, y), $(x : y) \in \mathbb{P}^1$. π is called the *algebraic Hopf map*, as the map on $\mathbb{C}$-points is homotopy equivalent to the classical Hopf map $\alpha : S^3 \to S^2$; the corresponding element of $\pi_3(S^2) \cong \mathbb{Z}$ is a generator, but in stable homotopy, this gives the generator of $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$ (the suspension $\Sigma\alpha : S^4 \to S^3$ is the generator of $\pi_4(S^3) = \mathbb{Z}/2$).

However, on the $\mathbb{R}$-points, $(\mathbb{A}^2 \setminus \{0\})(\mathbb{R}) = \mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to the S^1 $x^2 + y^2 = 1$, $\mathbb{P}^1(\mathbb{R}) = \mathbb{RP}^1$ is homeomorphic to S^1 and $(\mathbb{A}^2 \setminus \{0\})(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ is homotopy equivalent to the map $\times 2 : S^1 \to S^1$. Thus $Re_\sigma(\eta)$ induces $\times 2$ on the unit of SH.

One can express the involution τ in terms of η, in fact in $\mathrm{GW}(k) = \mathrm{End}(\mathbb{S}_k, \mathbb{S}_k)$ τ corresponds to the element $\langle -1 \rangle \in \mathrm{GW}(k)$ and thus

$$\tau = 1 + \eta[-1],\ (1/2)(1+\tau) = (1/2)h,\ (1/2)(1-\tau) = (-1/2)[-1]\eta.$$

Since τ acts by -1 on SH$(k)^-$, this tells us that η acts invertibly on SH$(k)^-$; in fact one has

$$\mathrm{SH}(k)^- \cong \mathrm{SH}(k)_\mathbb{Q}[\eta^{-1}].$$

For the plus part, Cisinski-Déglise [5, Equivalence (5.3.35.2), Theorem 16.1.4, Theorem 16.2.13] have described an equivalence of $\mathrm{SH}(k)^+$ with Voevodsky's triangulated category of motives over k (with $\mathbb{Q}$-coefficients)

$$\mathrm{SH}(k)^+ \cong \mathrm{DM}(k)_{\mathbb{Q}}$$

with the inclusion $\mathrm{SH}(k)^+ \subset \mathrm{SH}(k)_{\mathbb{Q}}$ corresponding to the "motivic Eilenberg-MacLane functor" $\mathrm{DM}(k) \to \mathrm{SH}(k)$. Since $Re_\sigma(\eta)_{\mathbb{Q}}$ induces $\times 2$ in $\mathrm{SH}_{\mathbb{Q}}$. As η acts by zero on $\mathrm{SH}(k)^+ = \mathrm{DM}(k)$, this tells us that $Re_{\sigma,\mathbb{Q}}$ is zero on $\mathrm{DM}(k)_{\mathbb{Q}}$ (in fact, Re_σ sends every morphism in $\mathrm{DM}(k)$ to a 2-torsion map in SH). As $\mathrm{SH}_{\mathbb{Q}}$ is equivalent to the derived category of $\mathbb{Q}$-vector spaces via the singular chain complex functor, this tells us that sending $X \in \mathbf{Sm}/k$ to $C_*(X(\mathbb{R})) \otimes \mathbb{Q} \in D(\mathbb{Q})$ extends to the functor $Re_{\sigma,\mathbb{Q}} : \mathrm{SH}(k) \to D(\mathbb{Q})$ and sends motives $\mathrm{DM}(k)$ to zero. This should explain the lack of any reasonable "periods for the real locus" in motives.

There is however a quadratic refinement of $\mathrm{DM}(k)$, due to Déglise and Fasel [3, 4], and constructed, roughly speaking, by replacing Voevodsky's category of finite "Chow" correspondences with a category of "finite Chow-Witt correspondences". This yields a quadratic refinement of motivic cohomology, with the $(2n, n)$ part giving the Chow-Witt groups. Let us denote this category by $\tilde{\mathrm{DM}}(k)$. There is a functor $\tilde{\mathrm{DM}}(k) \to \mathrm{SH}(k)$, and we thus have a real realization functor $Re_\sigma : \tilde{\mathrm{DM}}(k) \to D(\mathbf{Ab})$ by composition $\tilde{\mathrm{DM}}(k) \to \mathrm{SH}(k)$ with Re_σ and then the singular chain complex functor $\mathrm{SH} \to D(\mathbf{Ab})$.

Passing to $\mathbb{Q}$-coefficients, the above discussion suggests that we should localize with respect to η. This gives an equivalence

$$\tilde{\mathrm{DM}}(k)_{\mathbb{Q}}[\eta^{-1}] \cong \mathrm{SH}(k)_{\mathbb{Q}}[\eta^{-1}] \cong \mathrm{SH}(k)^-,$$

the real realization functor on $\tilde{\mathrm{DM}}(k)_{\mathbb{Q}}$ factors through $\tilde{\mathrm{DM}}(k)_{\mathbb{Q}}[\eta^{-1}]$, and thus can be understood by simply restricting $Re_{\sigma,\mathbb{Q}}$ to $\mathrm{SH}(k)^-$.

With Ananevsky and Panin, we have a fairly concrete description of the category $\mathrm{SH}(k)^-$ in terms of sheaves of $\mathcal{W}$-modules, where $\mathcal{W}$ is the sheaf of Witt rings on $\mathbf{Sm}/k_{\mathrm{Nis}}$. The sheaf $\mathcal{W}$ is a strictly $\mathbb{A}^1$-invariant Nisnevich sheaf on $\mathbf{Sm}/k$, meaning that the pull-back map

$$H^*_{\mathrm{Nis}}(X, \mathcal{W}) \to H^*_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{W})$$

is an isomorphism for all $X | in \mathbf{Sm}/k$. In addition, the identifications $\mathcal{W} \cong \mathcal{K}^{MW}_{-1} \cong \mathcal{K}^{MW}_{-2} \cong \ldots$ gives us a canonical isomorphism

$$\mathcal{W} \cong \mathcal{H}om(\mathbb{G}_m, \mathcal{W}).$$

For M a sheaf of $\mathcal{W}$-modules, the above isomorphism gives rise to a canonical map $\epsilon_M : M \to \mathcal{H}om(\mathbb{G}_m, M)$ via

$$M \cong M \otimes_{\mathcal{W}} \mathcal{W} \cong M \otimes_{\mathcal{W}} \mathcal{H}om(\mathbb{G}_m, \mathcal{W}) \to \mathcal{H}om(\mathbb{G}_m, M \otimes_{\mathcal{W}} \mathcal{W}) \cong \mathcal{H}om(\mathbb{G}_m, M).$$

We define the category of $\mathcal{W}$-motives $\mathrm{DM}_{\mathcal{W}}(k)$ (roughly speaking) as the full subcategory of the derived category of Nisnevich sheaves on $\mathbf{Sm}/k$, $D(Sh^{Nis}_{\mathbf{Ab}}(\mathbf{Sm}/k))$, with objects those complexes C such that the cohomology sheaves $\mathcal{H}^*(C)$ are strictly $\mathbb{A}^1$ invariant, and the map $\epsilon_C : C \to \mathcal{H}om(\mathbb{G}_m, C)$ is a quasi-isomorphism. We show [1, Theorem 4.2] that $\mathrm{DM}_{\mathcal{W}}(k)_{\mathbb{Q}}$ is equivalent to $\mathrm{SH}(k)^-$. We let $M_{\mathcal{W}}(X)$ be the object in $\mathrm{DM}_{\mathcal{W}}(k)$ corresponding to $\Sigma^\infty_T X_+$.

We raise the following question: for $\sigma : k \to \mathbb{R}$ an embedding, we have the real Betti realization $Re_\sigma : \mathrm{DM}_{\mathcal{W}}(k) \to D(\mathbb{Q})$. For $X \in \mathbf{Sm}/k$, we have the real de Rham complex $\mathcal{E}^*(X^\sigma(\mathbb{R}))$ of C^∞ $\mathbb{R}$-valued differential forms on $X(\mathbb{R})$. Is there an object $\Omega^{\mathcal{W}}$ in $\mathrm{DM}_{\mathcal{W}}(k)$ such that

1. For each $X \in \mathbf{Sm}/k$, there is a field k_X, finitely generated over k, and with $\mathrm{Hom}(M_{\mathcal{W}}(X), \Omega^{\mathcal{W}}[*])$ having a natural structure of graded k_X-vector space.
2. There is a natural isomorphism for $X \in \mathbf{Sm}/k$ and $\sigma : k_X \to \mathbb{R}$ an embedding

$$\mathrm{Hom}(M_{\mathcal{W}}(X), \Omega^{\mathcal{W}}[*]) \otimes_{k_X} \mathbb{R} \cong H^*(\mathcal{E}^*(X^\sigma(\mathbb{R}))).$$

If this were the case, one could then give the de Rham cohomology of $X^\sigma(\mathbb{R})$ a natural k_X-structure, and the classical comparison isomorphism

$$H^*(\mathcal{E}^*(X^\sigma(\mathbb{R}))) \cong H^*_{sing}(X^\sigma(\mathbb{R}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$$

would give rise to "real periods" for X as elements of $\mathbb{R}^\times/k_X^\times$.

In the case of the $\mathbb{C}$-realization $\mathrm{SH}(k) \to \mathrm{SH}$ associated to an embedding $\sigma : k \hookrightarrow \mathbb{C}$, the de Rham cohomology over k is represented by an object Ω in $\mathrm{SH}(k)_k$ (actually in $\mathrm{DM}(k)_k$). This was constructed by Ayoub: he takes the de Rham complex, considered as a complexes of sheaves Ω^*/k on $\mathbf{Sm}/k$. Multiplication with fundamental class $[\mathbb{P}^1] \in H^2_{dR}(\mathbb{P}^1)$ gives a quasi-isomorphism

$$\mathcal{H}om(T, \Omega^*/k[2]) \to \Omega^*/k$$

so by adjunction one gets maps $\epsilon_n : \Omega^*/k[2n] \wedge T \to \Omega^*/k[2n+2]$ and thereby a T-spectrum $\Omega \in \mathrm{SH}(k)_k$ representing de Rham cohomology. The k-structure on $\mathrm{Hom}_{\mathrm{SH}(k)_k}(\Sigma^\infty_T X_+, \Omega)$ gives a k-structure on $H^*(\mathcal{E}^*(X^\sigma(\mathbb{C}) \otimes \mathbb{C})$, that is, in the case of the $\mathbb{C}$-realization, one takes $k_X = k$. This is perhaps too much to ask for in the real setting.

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