Introduction

Using the Milnor-Witt $K$-sheaves as a “quadratic refinement’ of the Milnor $K$-sheaves, one can refine a number of constructions, such as intersection theory or top Chern classes of vector bundles, from the Chow ring to the \textit{Chow-Witt ring} (constructed by Barge-Morel [2] and studied in detail by Fasel [6] and others). In some cases, this allows one to refine the classical degree of an intersection or characteristic class to a degree with values in the Grothendieck-Witt ring $GW(k)$ of the base-field $k$. Examples of such refinements include the topological Euler characteristic as well as the Riemann-Hurwicz formula. There is also a conjectural relation with periods.

1. Milnor-Witt sheaves

For simplicity we work over a perfect base-field $k$ with $\text{char} k \neq 2$. We write $\text{Sm}/k$ for the category of smooth separated finite-type $k$-schemes.

For a field $F$, the Milnor $K$-theory of $F$ is the graded ring $K_M^*(F)$ defined by generators and relations:

- Generators: for $u \in F^\times$ one has the generator $\{u\}$ in degree +1.
- Relations:
  1. (Additivity) $\{uv\} = \{u\} + \{v\}$ for $u, v \in F^\times$.
  2. (Steinberg relation) $\{u\}\{1-u\} = 0$ for $u \in F \setminus \{0,1\}$

For a DVR $\mathcal{O}$ with quotient field $K$ and residue field $k$, there is a map $\partial : K_n^M(K) \to K_{n-1}^M(k)$, which allows one to define the unramified sheaf $\mathcal{K}_n^M$ on $\text{Sm}/k$ (for the Nisnevich topology). $\mathcal{K}_0^M$ is the constant sheaf $\mathbb{Z}$, $\mathcal{K}_1^M = \mathbb{G}_m$.

We have the Bloch/Kato formula relating the Chow ring with the cohomology of the Milnor sheaves:

$$\text{CH}^n(X) \cong H^n(X, \mathcal{K}_n^M).$$

For $f : Y \to X$ a morphism in $\text{Sm}/k$ we have the pull-back map $f^* : H^p(X, \mathcal{K}_n^M) \to H^p(Y, \mathcal{K}_n^M)$, which gives rise to the classical pull-back map $f^* : CH^p(X) \to CH^p(Y)$. If $f : Y \to X$ is proper of relative dimension $d$,}

\[ \]
there is a push-forward $f_* : H^p(Y, \mathcal{K}^M_{q}) \to H^{p-d}(X, \mathcal{K}^M_{q-d})$ which gives rise to the push-forward $f_* : \text{CH}^p(Y) \to \text{CH}^{p-d}(X)$.

Hopkins-Morel (see e.g. [11]) have defined the Milnor-Witt sheaf $K^{MW}_s$ as a “quadratic refinement” of the Milnor sheaf $K^M_s$. For a field $F$, $K^{MW}_s(F)$ is the graded ring defined by generators and relations:

- Generators: for $u \in F^\times$ one has the generator $[u]$ in degree +1. There is an additional generator $\eta$ in degree -1.
- Relations:
  1. (Twisted additivity) $[uv] = [u] + [v] + \eta[u][v]$ for $u, v \in F^\times$.
  2. (Steinberg relation) $[u][1-u] = 0$ for $u \in F \setminus \{0, 1\}$.
  3. (Hyperbolic relation) Let $h = 2 + \eta[-1]$. Then $\eta \cdot h = 0$.

For a DVR $\mathcal{O}$ with quotient field $K$, residue field $k$ and generator $t$ for the maximal ideal, there is a map $\partial_t : K^{MW}_n(K) \to K^{MW}_n(k)$. Although $\partial_t$ depends on $t$, this still gives rise to the unramified (Nisnevich) sheaf $K^{MW}_n$ on $\text{Sm}/k$.

Let $GW(F)$ denote the Grothendieck-Witt ring of non-degenerate quadratic forms over $F$: this is the group completion of the monoid (under orthogonal direct sum) of non-degenerate quadratic forms over $F$. The hyperbolic form is the rank 2 form $\langle x, y \rangle = x^2 - y^2$, and the Witt ring $W(F)$ is the quotient $GW(F)/\langle H \rangle$. Note that $\langle H \rangle = \mathbb{Z} \cdot H$, as for a quadratic form $q$ of rank $n$, one has $q \cdot H = n \cdot H$. For $u \in F^\times$ we let $q_u$ be the rank one form $q_u(x) = ux^2$.

For $u \in F^\times$ let $\langle u \rangle = 1 + \eta[u] \in K^{MW}_0(F)$.

**Theorem 1.1** (Morel [11, ]). Sending $q_u \in GW(F)$ to $\langle u \rangle \in K^{MW}_0(F)$ extends to an isomorphism of rings $GW(F) \to K^{MW}_0(F)$. Sending $q_u \in GW(F)$ to $\eta^n \langle u \rangle \in K^{MW}_{-n}(F)$ descends to an isomorphism of $GW(F)$-modules $W(F) \to K^{MW}_{-n}(F)$ for all $n \geq 1$.

Sending a unit $u$ to the element $\langle u \rangle$ gives a map of sheaves of abelian groups $\mathbb{G}_m \to K^{MW}_n$, which via the multiplication $K^{MW}_0 \times K^{MW}_n \to K^{MW}_n$ defines an action of $\mathbb{G}_m$ on $K^{MW}_n$. For a line bundle $L \to X$ on some $X \in \text{Sm}/k$, we have the $\mathbb{G}_m$-torsor $L^\times$ of nowhere zero sections of $L$ and we define the twisted Milnor-Witt sheaf as $K^{MW}_s(L) := K^{MW}_s \times_{\mathbb{G}_m} L^\times$. Note that there is a canonical isomorphism $K^{MW}_s(L \otimes M^{\otimes 2}) \cong K^{MW}_s(L)$, since $\langle u \rangle = \langle u^2 \rangle$. Morel’s isomorphism $\mathcal{K}^{MW}_0(F) \cong GW(F)$ extends to an identification of $K^{MW}_0(L)$ with $GW(L)$, the sheaf of $L$-valued quadratic forms (group completed as for GW).

Sending $\eta$ to zero defines a surjection $\pi : K^{MW}_n(L) \to K^M_n$ (of sheaves on $X$). As multiplication by $h$ kills $\eta$, this induces the hyperbolic map $h : K^M_n \to K^{MW}_n(L)$ and $\pi \circ h$ is multiplication by 2. For $n = 0$, the map $\pi : K^{MW}_0(L) \to K^M_0$ is the sheafified rank homomorphism.
Definition 1.2. Let $L \to X$ be a line bundle on some $X \in \text{Sm}_k$. Define the twisted Chow-Witt group $\tilde{\text{CH}}^n(X; L) := H^n(X, K_{MW}^{n}(L))$. The original definition of Barge-Morel is somewhat different, relying on combining Milnor $K$-theory and the powers of the augmentation ideal in $GW$, but subsequent developments show the two definitions yield the same groups.

For $f : Y \to X$ and $L \to X$ a line bundle, we have $f^* : H^p(X, K_{MW}^n(L)) \to H^p(Y, K_{MW}^n(f^*L))$, inducing $f^* : \tilde{\text{CH}}^n(X; L) \to \tilde{\text{CH}}^n(Y; f^*L)$. For $f : Y \to X$ proper of relative dimension $d$, we have $f_* : H^p(Y, K_{MW}^n(\omega_{Y/k} \otimes f^*L)) \to H^{p-d}(X, K_{MW}^{n-d}(\omega_{X/k} \otimes L))$, inducing $f_* : \tilde{\text{CH}}^n(Y; \omega_{Y/k} \otimes f^*L) \to \tilde{\text{CH}}^{n-d}(X, \omega_{X} \otimes L)$. See [2, 6] for details.

For $\pi_X : X \to \text{Spec} k$ a smooth and proper $k$-scheme of dimension $n$, we have the degree map

$\deg_k := \pi_{X*} : \text{CH}^n(X) \to \text{CH}^0(\text{Spec} k) = \mathbb{Z}$

and its quadratic refinement

$\tilde{\deg}_k := \pi_{X*} : \tilde{\text{CH}}^n(X; \omega_{X/k}) \to \tilde{\text{CH}}^0(\text{Spec} k) = GW(k)$.

2. Euler classes and Euler characteristic

Definition 2.1. Let $V \to X$ be a rank $r$ vector bundle on $X \in \text{Sm}_k$ with zero-section $s_0 : X \to V$. The Euler class $e(V) \in \tilde{\text{CH}}^r(X; \det^{-1}V)$ is defined as

$e(V) := s_0^*s_0(1_X)$

where $1_X \in \tilde{\text{CH}}^0(X) = H^0(X, \mathcal{O}_X)$ is the unit section.

For $X$ smooth and projective over $\text{Spec} k$, the quadratic Euler characteristic $\chi(X/k) \in GW(k)$ is defined as

$\chi(X/k) := \tilde{\deg}_k(e(T_{X/k}))$

where $T_{X/k} = \Omega_{X/k}^\vee$ is the tangent bundle.

Note that $e(T_{X/k})$ lives in $\tilde{\text{CH}}^{\dim_k X}(X; \omega_{X/k})$, so its quadratic degree is defined.

Here are some properties of the Euler class and Euler characteristic.

1. For $X \in \text{Sm}_k$ projective over $k$, $\text{rk}(\chi(X/k)) \in \mathbb{Z}$ is the “topological” Euler characteristic of $X$, which one can for example define as

$\chi^{\text{top}}(X) := \sum_{i=0}^{2\dim_k X} (-1)^i \dim_{\mathbb{Q}_\ell} H^i(\tilde{X}, \mathbb{Q}_\ell),$

where $\tilde{X} := X \times_k \bar{k}$, $\bar{k}$ an algebraic closure of $k$, and $\ell$ is a prime different from char$k$. If $k$ is a subfield of $\mathbb{C}$, one can also use singular cohomology of $X(\mathbb{C})$ to define $\chi^{\text{top}}(X)$, that is $\chi^{\text{top}}(X)$ is the usual topological Euler
characteristic of $X(\mathbb{C})$.

2. If we have a real embedding $\sigma : k \hookrightarrow \mathbb{R}$, $\sigma$ induces the signature map

$$\sigma^* : \text{GW}(k) \to \mathbb{Z}$$

by composing $\sigma_* : \text{GW}(k) \to \text{GW}(\mathbb{R})$ with the signature map $\text{sig} : \text{GW}(\mathbb{R}) \to \mathbb{Z}$. Then

$$\text{sig}^\sigma(\chi(X/k)) = \chi^{\text{top}}(\chi(\mathbb{R})).$$

3. For $X \in \text{Sm}/k$ projective over $k$, the $T$-suspension spectrum $\Sigma_{\infty}^T X_+$ in the motivic stable homotopy category $\text{SH}(k)$ (a symmetric monoidal category) is strongly dualizable [12, §2], hence there a well-defined trace

$$\text{Tr}(\text{Id}_{\Sigma_{\infty}^T X_+}) \in \text{End}_{\text{SH}(k)}(1 \text{SH}(k)).$$

By a theorem of Morel, there is a canonical identification

$$\text{End}_{\text{SH}(k)}(1 \text{SH}(k)) \cong \text{GW}(k)$$

and the corresponding element $\text{Tr}(\text{Id}_{\Sigma_{\infty}^T X_+}) \in \text{GW}(k)$ is the Euler characteristic $\chi(X/k)$. One can thus extend the definition of $\chi(X/k)$ to not necessarily proper $X \in \text{Sm}/k$ by setting $\chi(X/k) = \text{Tr}(\text{Id}_{\Sigma_{\infty}^T X_+})$ assuming that $\Sigma_{\infty}^T X_+$ is strongly dualizable. This is the case for all $X \in \text{Sm}/k$ if $k$ admits resolution of singularities, or in general if $X$ admits a smooth compactification $\bar{X} \supset X$ with $\bar{X} \setminus X$ a simple normal crossing divisor.

This interpretation gives us some additional properties of the Euler characteristic. A result of May [10, Theorem 0.1] says that in $\text{SH}(k)$, $\text{Tr}(\text{Id})$ is additive in distinguished triangles of strongly dualizable objects. Assuming for instance resolution of singularities for $k$, this gives us:

i. For $X = U \cup V$, $U, V \subset X$ open subschemes, we have

$$\chi(X/k) = \chi(U/k) + \chi(V/k) - \chi(U \cap V/k).$$

ii. Let $\pi : E \to X$ be a Zariski locally trivial bundle with fiber $F$ (and group $\text{Aut}(F)$). Then $\chi(E/k) = \chi(F/k) \cdot \chi(X/k)$.

iii. $\chi(\mathbb{P}^n) = \sum_{i=0}^{n}(-1)^i$, so if $\mathbb{P} \to X$ is a Zariski locally trivial $\mathbb{P}^n$-bundle, then

$$\chi(\mathbb{P}/k) = \chi(X/k) \cdot \sum_{i=0}^{n}(-1)^i.$$  

More generally, let $X \in \text{Sm}/k$ be a projective cellular variety: $X$ has a stratification by locally closed subsets $C_i$ with $C_i \cong \mathbb{A}^n$, then $\text{CH}^n(X)$ is a free abelian group of finite rank for each $n$. Let $r_{\text{ev}} = \sum_i \text{rk} \text{CH}^{2i}(X)$, $r_{\text{odd}} = \sum_i \text{rk} \text{CH}^{2i+1}(X)$, then

$$\chi(X/k) = \sum_{j=0}^{\dim X} (\text{rk} \text{CH}^i(X)) \cdot (-1)^j = \sum_{j=1}^{r_{\text{ev}}} x_j^2 + \sum_{j=1}^{r_{\text{odd}}} (-1) \cdot y_j^2.$$  

iv. Let $X = \text{Spec} F$, with $F \supset k$ a finite separable extension. Then $\chi(X/k)$ is the trace form $\text{Tr}_{F/k}(1)$.

v. Let $Z \subset X$ be a smooth codimension $r$ closed subscheme, $\bar{X} \to X$ the
blow-up of $X$ along $Z$. Then $\chi(\tilde{X}/k) = \chi(X/k) + \chi(Z) \cdot \sum_{i=1}^{r} (-1)^i$.

4. For $V \to X$ a rank $r$ vector bundle with dual bundle $V^\vee$, $e(V) = \langle -(-1)^i e(V^\vee) \rangle$.

5. If $V \to X$ is an odd rank bundle, then $\eta e(V) = 0$ in $H^r(X, K_r^{MW}(\det^{-1} V))$. In consequence, if $X$ is smooth and projective of odd dimension over $k$, then $\chi(X/k) = N \cdot H$ for some integer $N$.

6. Let $V \to X$ be a rank $r$ vector bundle and let $L \to X$ be a line bundle such that $L \cong M^\otimes 2$ for some line bundle $M$ on $X$. Then

\begin{equation}
(2.1) \quad e(V \otimes L) = e(V) + \sum_{i=1}^{r} c_1(M) \cdot c_{r-i}(V) \cdot c_1(L)^{i-1} \cdot h
\end{equation}

where $c_1(M), c_1(L)$ and the $c_{r-i}(V)$ are all in $\text{CH}^*(X)$, $c_0(V) := 1$, and $(-) \cdot h$ is the hyperbolic map $(-) \cdot h : H^r(X, K_r^M) \to H^r(X, K_r^{MW}(\det^{-1} V))$.

See [9] for details. In their paper [7] Kass and Wickelgren define a class in $GW(k)$ associated to a vector bundle (with some restrictions) on a smooth projective variety, and use this to get an interesting quadratic invariant of the “count” of lines on a smooth cubic surface in $\mathbb{P}^3_k$. It turns out that their invariant is exactly the quadratic degree of the Euler class of a vector bundle $V$, assuming the necessary condition on $\det V$ so that the quadratic degree of $e(V)$ is defined.

3. Local Euler classes

Let $V \to X$ be a rank $r$ vector bundle with zero section $0_V \subset V$. There is a Thom class $th(V) \in H^r_{0_V}(V, K_r^{MW}(\det^{-1} V))$. Let $s : X \to V$ be a section and $Z \subset X$ a closed subset with supp $(s) \subset Z$. We define the Euler class with supports $e_Z(V; s) \in H^r_Z(X, K_r^{MW}(\det^{-1} V))$ as $e_Z(V; s) = s^* (th(V))$.

Suppose that $\text{rank} V = \dim_k X = n$ and that supp $(s) = \{x_1, \ldots, x_r\}$ is a finite set of closed points of $X$. As

\[H^n_{\{x_1, \ldots, x_r\}}(X, K_n^{MW}(\det^{-1} V)) = \bigoplus_{i=1}^{r} H^n_{x_i}(X, K_n^{MW}(\det^{-1} V)),\]

we can write

\[e_{\{x_1, \ldots, x_r\}}(V; s) = \sum_{i=1}^{r} e_{x_i}(V; s); \quad e_{x_i}(V; s) \in H^n_{x_i}(X, K_n^{MW}(\det^{-1} V)).\]

Kass and Wickelgren [8] have developed an explicit formula for the local class $e_x(V; s), x = x_i$, as follows. For a line bundle $L$ on $X$, $n = \dim_k X$, there is a purity isomorphism

\[H^n_x(X, K_n^{MW}(L)) \cong GW(k(x); L \otimes \omega_{X/k}^{-1} \otimes k(x))\]
where $GW(k(x); L \otimes \omega_{X/k} \otimes k(x))$ is the Grothendieck-Witt group of $L \otimes \omega_{X/k} \otimes k(x)$-valued quadratic forms.

Choose parameters $t_1, \ldots, t_n \in O_{X,x}$ and a local basis $e_1, \ldots, e_n$ of sections for $V$ near $x$. Then $s = \sum_{i=1}^n s_i e_i$ with $s_i \in m_x \subset O_{X,x}$. Write $s_i = \sum_{j=1}^n a_{ij} t_j$, $a_{ij} \in O_{X,x}$, let $Q(s) = O_{X,x}/(s_1, \ldots, s_n)$ and let $E$ be the image in $Q(s)$ of $\det(a_{ij})$. Then $Q(s)$ is a $k(x)$-algebra, $(E)$ is the socle of $Q(s)$, $(E) \cong k(x)$. Let $\varphi : Q(s) \to k(x)$ be a $k(x)$-linear map with $\varphi(E) = 1$, and let

$$q_\varphi : Q(s) \to \det^{-1} V \otimes \omega^{-1}_{X/k} \otimes k(x)$$

be the quadratic form

$$q_\varphi(y) = \varphi(y^2) \cdot (e_1 \wedge \ldots \wedge e_n)^{-1} \cdot (dt_1 \wedge \ldots \wedge dt_n)^{-1}.$$

Then under the purity isomorphism $e_x(V, s)$ maps to $q_\varphi \in GW(k(x); \det^{-1} V \otimes \omega^{-1}_{X/k} \otimes k(x))$.

4. RIEMANN-HURWITZ FORMULA

Theorem 4.1 ([9, Corollary 12.3]). Let $X \in \text{Sm}/k$ be projective of dimension $n$, $C \in \text{Sm}/k$ a projective curve and $f : X \to C$ a separable map, giving us the non-zero section $df \in H^0(X, \Omega_{X/k} \otimes f^* \omega_{C/k})$. We suppose that there is a line bundle $M$ on $C$ with $\omega_{C/k} \cong M^\otimes 2$ and that $df$ has only the isolated zeros $x_1, \ldots, x_r$. Then

$$\chi(X/k) = (-1)^n \cdot \sum_{i=1}^r \text{Tr}_{k(x_i)/k} e_{x_i}(\Omega_{X/k} \otimes f^* \omega_{C/k}^{-1}, df)$$

$$- \frac{(-1)^n}{2} \deg_k (c_{n-1}(\Omega_{X/k}) \cdot f^* c_1(\omega_{C/k})) \cdot H$$

Here we use a chosen isomorphism $\omega_{C/k} \cong M^\otimes 2$ to view $e_{x_i}(\Omega_{X/k} \otimes f^* \omega_{C/k}^{-1}, df) \in GW(k(x); f^* \omega_{C/k})$ as an element of $GW(k(x))$, and $\text{Tr}_{k(x_i)/k}$ is the trace map $\text{Tr}_{k(x_i)/k} : GW(k(x_i)) \to GW(k)$.

Bemerkung. If $k$ is not perfect, the result still holds if one assumes that each residue field $k(x_i)$ is separable over $k$.

5. SOME EXAMPLES

Curves

Let $f : X \to \mathbb{P}^1_k$ be a separable map with $X$ a smooth projective geometrically integral curve over $k$ of genus $g_X$. Since $X$ has odd dimension, $\chi(X/k)$ is a multiple of $H$ and since $\text{rnk}(\chi(X/k)) = \chi^{top}(X) = 2 - 2g_X$, we have

$$\chi(X/k) = (1 - g_X) \cdot H.$$ Suppose that $f$ is ramified at points $x_1, \ldots, x_r \in X$, and let $y_i = f(x_i) \in \mathbb{P}^1$. Without loss of generality we may assume that the $y_i$ are all in $\mathbb{P}^1 \setminus \{0\} = \text{Spec} k[T]$. Let $g_i(T) \in k[T]$ be the monic irreducible equation for. $y_i \in$
we restrict to the case of even \( n \) hypersurface.

Taking ranks (noting that \( \text{rnk} \) addition to the classical Riemann-Hurwitz formula, this yields for some integer \( m \) to \( GW(\text{df}) \), local leading terms of and thus the quadratic Riemann-Hurwitz formula yields a relation on the

where we use the isomorphism \( k = \text{Spec} O \) is a parameter in \( \mathbb{A}^1 \) to \( dT \).

Chooses a parameter \( t_i \in O_{X,x_i} \). Then \( f^*(ds_i) = v_it_i^{e_i}dt_i \) for an integer \( e_i \geq 2 \) and a unit \( v_i \in O_{X,x_i} \). Let \( \bar{v}_i \) denote the image of \( v_i \) in \( k(x_i) \). The Kass-Wickelgren formula gives

\[
e_{x_i}(\Omega_{X/k} \otimes f^*\omega_{C/k}^{-1}; \text{df}) = \langle \bar{v}_i \rangle \cdot \sum_{i=0}^{e_i-2} (-1)^i \in GW(k(x_i))
\]

where we use the isomorphism \( \omega_{\mathbb{P}^1/k} \cong O_{\mathbb{P}^1}(1)\otimes^2 \) to pass from \( GW(k(x_i); f^*\omega_{\mathbb{P}^1/k}^{-1}) \) to \( GW(k(x_i)) \). Our Riemann-Hurwitz formula then gives

\[
(g_X - 1) \cdot H = (-\langle -1 \rangle) \cdot \chi(X/k) = \sum_{i=1}^r \left( \text{Tr}_{k(x_i)/k}(\bar{v}_i) \cdot \sum_{i=0}^{e_i-2} (-1)^i \right) + \text{deg} f \cdot H
\]

Taking ranks (noting that \( \text{rnk} H = 2 \)) recovers the classical Riemann-Hurwitz formula.

We note that

\[
\langle \bar{v}_i \rangle \cdot \sum_{i=0}^{e_i-2} (-1)^i = \begin{cases} \frac{e_i-1}{2} \cdot H & \text{for } e_i \text{ odd}, \\ \langle \bar{v}_i \rangle + \frac{e_i-2}{2} \cdot H & \text{for } e_i \text{ even}, \\ \end{cases}
\]

and thus the quadratic Riemann-Hurwitz formula yields a relation on the local leading terms of \( df \):

\[
\sum_{i,e_i \text{ even}} \text{Tr}_{k(x_i)/k}(\bar{v}_i) = m \cdot H
\]

for some integer \( m \).

In case \( \text{char} k = 0 \), or more generally if all the ramification in \( f : X \to \mathbb{P}^1_k \) is tame, we have \( f^*(s_i) = u_it_i^{e_i} \) for a unit \( u_i \in O_{X,x_i}^\times \) and \( v_i = e_iu_i \).

In addition to the classical Riemann-Hurwitz formula, this yields

\[
\sum_{i,e_i \text{ even}} \text{Tr}_{k(x_i)/k}(e_i\bar{v}_i) = m \cdot H
\]

for some integer \( m \).

**Diagonal hypersurfaces**

Up to now in all our examples \( \chi(X/k) \) is either hyperbolic, or hyperbolic plus \( (1) \). We can also compute \( \chi(X/k) \) for \( X \subset \mathbb{P}^{n+1}_k \) a degree \( m \) diagonal hypersurface, that is, a smooth hypersurface defined by degree \( m \) polynomial \( \sum_{i=0}^{n+1} a_iT^m \). As \( \chi(X/h) \) is hyperbolic if \( n \) is odd, we restrict to the case of even \( n \).
All smooth hypersurfaces of fixed degree \( m \) in \( \mathbb{P}^{n+1}_k \) have the same topological Euler characteristic \( \chi^{top}(X) \). We let

\[
A_{n,m} = \begin{cases} 
\frac{1}{2}(\chi^{top}(X) - 1) & \text{for } m \text{ odd,} \\
\frac{1}{2}(\chi^{top}(X) - 2) & \text{for } m \text{ even.}
\end{cases}
\]

**Theorem 5.1** ([9, ]). Let \( X \subset \mathbb{P}^{n+1}_k \) be a smooth degree \( m \) hypersurface with defining equation \( \sum_{i=0}^{n+1} a_i T^m = 0 \) and with \( n \) even. Then \( A_{n,m} \) is an integer and

\[
\chi(X/k) = \begin{cases} 
A_{n,m} \cdot H + \langle m \rangle & \text{for } m \text{ odd,} \\
A_{n,m} \cdot H + \langle m \rangle + \langle -m \cdot \prod_{i=0}^{n+1} a_i \rangle & \text{for } m \text{ even.}
\end{cases}
\]

The idea of the proof is to take \( Z \subset X \) to be the closed subscheme defined by \( T_n = T_{n+1} = 0 \) and let \( \tilde{X} \) to be the blow-up of \( X \) along \( Z \). Then \( Z \subset \mathbb{P}^{n-1}_k \) is the diagonal hypersurface with defining equation \( \sum_{i=0}^{n-1} a_i T^m = 0 \), so we can use induction to compute \( \chi(Z/k) \). The blow-up formula

\[
\chi(\tilde{X}/k) = \chi(X/k) + (-1)^n \chi(Z/k)
\]
tells us it suffices to compute \( \chi(\tilde{X}/k) \). We have the morphism

\[
f := (T_n : T_{n+1}) : \tilde{X} \to \mathbb{P}^1
\]

which has as isolated critical points the closed subscheme of \( \tilde{X} \) (or \( X \)) defined by \( T_0 = \ldots = T_{n-1} = 0 \), i.e. the closed subscheme \( x \) of \( \mathbb{P}^{n+1}_k \) defined by \( T_0 = \ldots = T_{n-1} = 0 \), \( a_n T^m + a_{n_1} T_{n+1} = 0 \). One then uses the Kass-Wickelgren formula to compute the local term \( e_x(\Omega_{(X/k)} \otimes f^* \omega_{\mathbb{P}^1/k}, df) \) and then the Riemann-Hurwitz formula computes \( \chi(\tilde{X}/k) \).

### 6. Some thoughts on the real realization

We write \( (X, x) \) for a \( k \)-scheme \( X \) pointed by an \( x \in X(k) \). Set \( G_m = (\mathbb{A}^1 \setminus \{0\}, \{1\}) \), \( T = (\mathbb{P}^1, \infty) \), \( X_+ = (X \amalg \text{Spec } k, \text{Spec } k) \). We let \( \text{SH}(k) \) denote the motivic stable homotopy category and \( \text{SH} \) the classical topological stable homotopy category. Objects of \( \text{SH} \) are spectra, that is, sequences of pointed spaces \( (E_0, E_1, \ldots, E_n, \ldots) \), together with maps \( \epsilon_n : \Sigma E_n \to E_{n+1} \), \( \Sigma E_n := E_n \land S^1 \). Objects of \( \text{SH}(k) \) are \( T \)-spectra \( (T = (\mathbb{P}^1, \infty)) \), that is, sequences \( (E_0, E_1, \ldots, E_n, \ldots) \) with each \( E_n \) a point presheaf of spaces on \( \text{Sm}/k \), together with maps \( \epsilon_n : \Sigma T E_n \to E_{n+1} \), \( \Sigma T E_n := E_n \land T \). The suspension spectrum \( \Sigma^\infty T X_+ \in \text{SH}(k) \) associated to an \( X \in \text{Sm}/k \) is the sequence \( (X_+, X_+ \land T, \ldots, X_+ \land T^\land n, \ldots) \) with identity maps \( \epsilon_n \), just as the classical suspension spectrum \( \Sigma^\infty X_+ \) of a space \( X \) is the the sequence \( (X_+, X_+ \land S^1, \ldots, X_+ \land S^m, \ldots) \) with identity maps \( \epsilon_n \). Suppose we have an embedding \( \sigma : k \to \mathbb{R} \). Sending \( X \in \text{Sm}/k \) to the space of \( \mathbb{R} \)-points \( X(\mathbb{R}) \) extends to a **real realization**

\[
\text{Re}_{\sigma} : \text{SH}(k) \to \text{SH},
\]
where \( \text{SH} \) is the classical stable homotopy category of spectra and \( R e_\eta (\Sigma_T^\infty X_+) = \Sigma^\infty X(\mathbb{R})_+ \).

We recall that the unit in \( \text{SH}(k) \) is the motivic sphere spectrum

\[
S_k := \Sigma^\infty T \text{Spec} \, k = (\text{Spec} \, k, T, T^{\wedge 2}, \ldots).
\]

The involution on \( T \wedge T \) exchanging the two factors defined an involution \( \tau : S_k \to S_k \), and as \( \text{SH}(k) \) idempotently complete, we have a direct sum “eigenspace” decomposition in \( \text{SH}(k)[1/2] \)

\[
S_k = S_k^+ \oplus S_k^-\]

corresponding to writing \( \text{Id}_{S_k} = (1/2)(\text{Id}_{S_k} - \tau) + (1/2)(\text{Id}_{S_k} + \tau) \). As \( S_k \) is the unit in \( \text{SH}(k) \), this gives the decomposition of \( \text{SH}(k)_{Q} \) as

\[
\text{SH}(k)_{Q} = \text{SH}(k)^+ \times \text{SH}(k)^-\]

where \( \tau \) acts by \( \text{Id} \) on \( \text{SH}(k)^+ \) and by \( -\text{Id} \) on \( \text{SH}(k)^- \).

We have already mentioned Morel’s theorem, that \( \text{End}_{\text{SH}(k)}(S_k) \cong \text{GW}(k) \cong K_0^{MW}(k) \); in fact Morel has computed (here \( n, m \geq 0 \))

\[
\text{Hom}_{\text{SH}(k)}(\Sigma_T^\infty \mathbb{G}_m^\wedge m, \Sigma_T^\infty \mathbb{G}_m^{n+m}) \cong K_n^{MW}(k)
\]

and for \( n > 0 \),

\[
\text{Hom}_{\text{SH}(k)}(\Sigma_T^\infty \mathbb{G}_m^{n+m}, \Sigma_T^\infty \mathbb{G}_m^\wedge m) \cong K_{-n}^{MW}(k) \cong W(k).
\]

In particular, there is an element in

\[
\text{Hom}_{\text{SH}(k)}(\Sigma_T^\infty \mathbb{G}_m^\wedge 2, \Sigma_T^\infty \mathbb{G}_m) = \text{Hom}_{\text{SH}(k)}(\Sigma_T^\infty \mathbb{S}^1 \wedge \mathbb{G}_m^\wedge 2, \Sigma_T^\infty \mathbb{S}^1 \wedge \mathbb{G}_m)
\]

corresponding to the degree -1 generator \( \eta \in K_{-1}^{MW}(k) \).

This map has a very simple geometric description. In the unstable motivic homotopy category \( \mathcal{H}_*(k) \) there are canonical isomorphisms

\[
T \cong (\mathbb{P}^1, 1) \cong \mathbb{S}^1 \wedge \mathbb{G}_m, (A^2 \setminus \{0\}, (1, 1)) \cong \mathbb{S}^1 \wedge \mathbb{G}_m^\wedge 2
\]

and \( \eta \) is the stabilization of the map \( \pi : A^2 \setminus \{0\} \to \mathbb{P}^1 \) sending \((x, y) \in A^2 \setminus \{0\} \) to the line through \((x, y), (x : y) \in \mathbb{P}^1 \). \( \pi \) is called the algebraic Hopf map, as the map on \( \mathbb{C} \)-points is homotopy equivalent to the classical Hopf map \( \alpha : S^3 \to S^2 \); the corresponding element of \( \pi_3(S^2) \cong \mathbb{Z} \) is a generator, but in stable homotopy, this gives the generator of \( \pi_1(\mathbb{S}) \cong \mathbb{Z}/2 \) (the suspension \( \Sigma \alpha : S^4 \to S^3 \) is the generator of \( \pi_3(S^3) = \mathbb{Z}/2 \).

However, on the \( \mathbb{R} \)-points, \((A^2 \setminus \{0\})(\mathbb{R}) = \mathbb{R}^2 \setminus \{0\}\) is homotopy equivalent to the \( S^1 \) \( x^2 + y^2 = 1 \), \( \mathbb{P}^1(\mathbb{R}) = \mathbb{RP}^1 \) is homeomorphic to \( S^1 \) and \((A^2 \setminus \{0\})(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})\) is homotopy equivalent to the map \( x \times 2 : S^1 \to S^1 \). Thus \( R e_\eta(\eta) \) induces \( x \times 2 \) on the unit of \( \text{SH} \).

One can express the involution \( \tau \) in terms of \( \eta \), in fact in \( \text{GW}(k) = \text{End}(S_k, S_k) \) \( \tau \) corresponds to the element \((−1) \in \text{GW}(k) \) and thus

\[
\tau = 1 + \eta[−1], \quad (1/2)(1 + \tau) = (1/2)h, \quad (1/2)(1 − \tau) = (−1/2)[−1]\.eta.
\]

Since \( \tau \) acts by \(-1\) on \( \text{SH}(k)^− \), this tells us that \( \eta \) acts invertibly on \( \text{SH}(k)^− \); in fact one has

\[
\text{SH}(k)^− \cong \text{SH}(k)_{Q}[\eta^{-1}]\]

For the plus part, Cisinski-Déglise [5, Equivalence (5.3.35.2), Theorem 16.1.4, Theorem 16.2.13] have described an equivalence of $\text{SH}(k)^+ \cong \text{DM}(k)_{\mathbb{Q}}$

with the inclusion $\text{SH}(k)^+ \subset \text{SH}(k)_{\mathbb{Q}}$ corresponding to the “motivic Eilenberg-MacLane functor” $\text{DM}(k) \to \text{SH}(k)$. Since $\text{Re}_\sigma(\eta)_{\mathbb{Q}}$ induces $\times 2$ in $\text{SH}_{\mathbb{Q}}$. As $\eta$ acts by zero on $\text{SH}(k)^+ = \text{DM}(k)$, this tells us that $\text{Re}_\sigma, \mathbb{Q}$ is zero on $\text{DM}(k)_{\mathbb{Q}}$ (in fact, $\text{Re}_\sigma$ sends every morphism in $\text{DM}(k)$ to a 2-torsion map in $\text{SH}$). As $\text{SH}_{\mathbb{Q}}$ is equivalent to the derived category of $\mathbb{Q}$-vector spaces via the singular chain complex functor, this tells us that sending $X \in \text{Sm}/k$ to $C_*(X(\mathbb{R})) \otimes \mathbb{Q} \in D(\mathbb{Q})$ extends to the functor $\text{Re}_\sigma, \mathbb{Q} : \text{DM}(k) \to D(\mathbb{Q})$ and sends motives $\text{DM}(k)$ to zero. This should explain the lack of any reasonable “periods for the real locus” in motives.

There is however a quadratic refinement of $\text{DM}(k)$, due to Déglise and Fasel [3, 4], and constructed, roughly speaking, by replacing Voevodsky’s category of finite “Chow” correspondences with a category of “finite Chow-Witt correspondences”. This yields a quadratic refinement of motivic cohomology, with the $(2n, n)$ part giving the Chow-Witt groups. Let us denote this category by $\tilde{\text{DM}}(k)$. There is a functor $\tilde{\text{DM}}(k) \to \text{SH}(k)$, and we thus have a real realization functor $\text{Re}_\sigma : \tilde{\text{DM}}(k) \to D(\text{Ab})$ by composition $\tilde{\text{DM}}(k) \to \text{SH}(k)$ with $\text{Re}_\sigma$ and then the singular chain complex functor $\text{SH} \to D(\text{Ab})$.

Passing to $\mathbb{Q}$-coefficients, the above discussion suggests that we should localize with respect to $\eta$. This gives an equivalence

$$\text{DM}(k)_{\mathbb{Q}}[\eta^{-1}] \cong \text{SH}(k)_{\mathbb{Q}}[\eta^{-1}] \cong \text{SH}(k)^{-}$$

the real realization functor on $\text{DM}(k)_{\mathbb{Q}}$ factors through $\tilde{\text{DM}}(k)_{\mathbb{Q}}[\eta^{-1}]$, and thus can be understood by simply restricting $\text{Re}_\sigma, \mathbb{Q}$ to $\text{SH}(k)^{-}$.

With Ananlevsky and Panin, we have a fairly concrete description of the category $\text{SH}(k)^{-}$ in terms of sheaves of $W$-modules, where $W$ is the sheaf of Witt rings on $\text{Sm}/k_{\text{Nis}}$. The sheaf $W$ is a strictly $A^1$-invariant Nisnevich sheaf on $\text{Sm}/k$, meaning that the pull-back map

$$H^*_\text{Nis}(X, W) \to H^*_\text{Nis}(X \times A^1, W)$$

is an isomorphism for all $X \in \text{Sm}/k$. In addition, the identifications $W \cong K^{-1}_W \cong K^{-2}_W \cong \ldots$ gives us a canonical isomorphism $W \cong \text{Hom}(\mathbb{G}_m, W)$.

For $M$ a sheaf of $W$-modules, the above isomorphism gives rise to a canonical map $\epsilon_M : M \to \text{Hom}(\mathbb{G}_m, M)$ via

$$M \cong M \otimes_W W \cong M \otimes_W \text{Hom}(\mathbb{G}_m, W)$$

$$\quad \to \text{Hom}(\mathbb{G}_m, M \otimes_W W) \cong \text{Hom}(\mathbb{G}_m, M).$$
We define the category of \( \mathcal{W} \)-motives \( DM_\mathcal{W}(k) \) (roughly speaking) as the full subcategory of the derived category of Nisnevich sheaves on \( \text{Sm}/k \), \( D(\text{Sh}^{\text{Nis}}_{\text{Ab}}(\text{Sm}/k)) \), with objects those complexes \( C \) such that the cohomology sheaves \( \mathcal{H}^*(C) \) are strictly \( k \)-invariant, and the map \( \epsilon_C : C \to \mathcal{H}om(\mathbb{G}_m, C) \) is a quasi-isomorphism. We show \([1, \text{Theorem 4.2}]\) that \( DM_\mathcal{W}(k)_Q \) is equivalent to \( \text{SH}(k)_- \). We let \( M_\mathcal{W}(X) \) be the object in \( DM_\mathcal{W}(k) \) corresponding to \( \Sigma^\infty_+ X_- \).

We raise the following question: for \( \sigma : k \to \mathbb{R} \) an embedding, we have the real Betti realization \( R_{\text{re}} : DM_\mathcal{W}(k) \to D(\mathbb{Q}) \). For \( X \in \text{Sm}/k \), we have the real de Rham complex \( \mathcal{E}^*(X^\sigma(\mathbb{R})) \) of \( C^\infty \mathbb{R} \)-valued differential forms on \( X(\mathbb{R}) \). Is there an object \( \Omega^\mathcal{W} \) in \( DM_\mathcal{W}(k) \) such that

1. For each \( X \in \text{Sm}/k \), there is a field \( k_X \), finitely generated over \( k \), and with \( \text{Hom}(M_\mathcal{W}(X), \Omega^\mathcal{W}[\ast]) \) having a natural structure of graded \( k_X \)-vector space.
2. There is a natural isomorphism for \( X \in \text{Sm}/k \) and \( \sigma : k_X \to \mathbb{R} \) an embedding

\[
\text{Hom}(M_\mathcal{W}(X), \Omega^\mathcal{W}[\ast]) \otimes_{k_X} \mathbb{R} \cong H^*(\mathcal{E}^*(X^\sigma(\mathbb{R}))).
\]

If this were the case, one could then give the de Rham cohomology of \( X^\sigma(\mathbb{R}) \) a natural \( k_X \)-structure, and the classical comparison isomorphism

\[
H^*(\mathcal{E}^*(X^\sigma(\mathbb{R}))) \cong H^*_\text{sing}(X^\sigma(\mathbb{R}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}
\]

would give rise to “real periods” for \( X \) as elements of \( \mathbb{R}^X/k_X^\times \).

In the case of the \( \mathbb{C} \)-realization \( \text{SH}(k) \to \text{SH} \) associated to an embedding \( \sigma : k \hookrightarrow \mathbb{C} \), the de Rham cohomology over \( k \) is represented by an object \( \Omega \) in \( \text{SH}(k)_k \) (actually in \( DM(k)_k \)). This was constructed by Ayoub: he takes the de Rham complex, considered as a complexes of sheaves \( \Omega^*/k \) on \( \text{Sm}/k \). Multiplication with fundamental class \([P^1] \in H^2(\mathbb{P}^1)\) gives a quasi-isomorphism

\[
\text{Hom}(T, \Omega^*/k[2]) \to \Omega^*/k
\]

so by adjunction one gets maps \( \epsilon_n : \Omega^*/k[2n] \wedge T \to \Omega^*/k[2n + 2] \) and thereby a \( T \)-spectrum \( \Omega \in \text{SH}(k)_k \) representing de Rham cohomology. The \( k \)-structure on \( \text{Hom}_{\text{SH}(k)_k}(\Sigma^\infty_+ \Omega, \Omega) \) gives a \( k \)-structure on \( H^*(\mathcal{E}^*(X^\sigma(\mathbb{C}) \otimes \mathbb{C})) \), that is, in the case of the \( \mathbb{C} \)-realization, one takes \( k_X = k \). This is perhaps too much to ask for in the real setting.

References


