The Relational Complexity of a Finite Permutation Group

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HIM, 10/2018

Structures and Permutation Groups

Structure		*Permutation Group*
A	$\rightarrow$	$\mathrm{Aut}(A)$ on A
$\mathrm{Inv}(G)$	$\leftarrow$	G on A

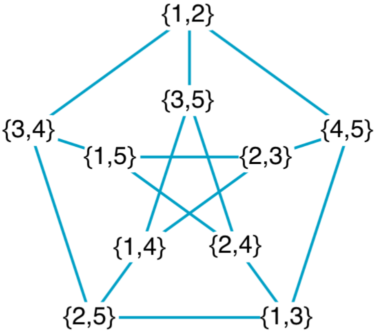
$$\mathrm{Inv}(G) = \bigcup A^n/G.$$

L_k: A^k/G (k-types) *language*

- *k-closed:* $G = \mathrm{Aut}(L_k)$
- *k-homogeneous:* A^ℓ-orbits are determined by L_k-isomorphism, all ℓ.

k-homogeneous $\implies$ G^X k-closed, all subsets X.

Example: Petersen Graph



2-closed

3-homogeneous: Independent triples are triangles or stars

Relational complexity 3.

Examples: Homogeneous graphs

(Gardiner, Sheehan)

$m \cdot K_n \quad C_5\ (D_{2\cdot 5}) \quad [3]^2\ (\mathrm{Sym}_3 \wr \mathrm{Sym}_2)$

What about C_n?
Metrically homogeneous, complexity 2, but $|A^2/G| \to \infty$.
(Bounded complexity, unbounded language)
What about $[n]^2$?
For $n \geq 4$: relational complexity is 4. (Edges $||$ or $\perp$)

Relational complexity spectrum: $[2, 4]$ (mind the gap).

Relational complexity spectrum

$\mathbf{a} \overset{r-1}{\sim} \mathbf{a}'$; $\mathbf{a} \not\sim \mathbf{a}'$; length r.
W.l.o.g. $\mathbf{a}, \mathbf{a}'$ agree on $r-1$ entries

$$(a_1, \ldots, a_{r-1}, a)$$
$$(a_1, \ldots, a_{r-1}, a')$$

Example ($\mathrm{GL}(n, q)$, $q > 2$)

$(v_1, \ldots, v_n, v)$ and $(v_1, \ldots, v_n, v')$ basis and two linear combinations with non-zero coefficients.
Relational complexity $n + 1$.

Example: $AO(n, F)$ anisotropic

Affine space with an anisotropic quadratic form

Relational complexity 2

$$Q(x_i - x_j) \text{ all } i, j$$

$$|A^{(2)}/G| = q - 1 \text{ (edge colors)}$$

$AO^-(2,3) = [3]^2$

$AO^-(2,4) =$ "Gleason graph"

Some issues

- Bounded relational complexity $\implies$??
- Compute or estimate relational complexity for “natural” actions
- Meaning of gaps in the relational complexity spectrum.
- Determine a canonical language.

Relational complexity 2

Conjecture

If G is binary and primitive then it is one of the following.

- $\mathbb{Z}/p\mathbb{Z}$ *acting regularly*
- $AO(n, q)$ *anisotropic (so* $n \leq 2$*)*
- $\mathrm{Sym}(n)$*, acting naturally.*

Ch., Wiscons: reduction to almost simple case

Gill, Spiga, Dalla Volta, Hunt, Liebeck: ongoing
— Done: Alternating socle; Lie rank 1; sporadic

Upper bounds

Stabilizer height:

Breadth of the lattice of pointwise stabilizers; $\min r$ so that

any G_A is $G_{A'}$ for some $A' \subseteq A$, $|A'| \leq r$

Nondegenerate sequence:

Independent point stabilizers (e.g., minimal base).
Stabilizer height tends to be maximal size of minimal base.

Some exceptions:
$\mathrm{P\Gamma U}(3,4)$ in degree 65, $3^4 : 16$, or J_2 on Hall-Janko graph

Relational complexity is bounded by stabilizer height plus 1
(so orbit tree search terminates)

Example (Revisited)

$\mathrm{GL}_n(q)$: stabilizer height n, relational complexity $n+1$ unless $q = 2$.

More upper bounds

• For a finite abelian group A, the maximal relational complexity in any action is rank(A)+1.
• For $H \triangleleft G$ with abelian quotient, the relational complexity is bounded by

$$\text{Stabilizer height of } H + \text{rank of } G/H + 1$$

Example

The relational complexity of $A\Gamma L(n, q)$ is at most $n + 3$

Hence for $q > 2$, it is $n + 2$ or $n + 3$.

Problem

When does $A\Gamma L_1(q)$ *have relational complexity* 3*?* 4*?*

Action on k-sets

Example

Stabilizer height of Sym_n on k-sets

$$n - 1 \ (k = 1)$$
$$n - 2 \ (k = 2 \text{ or } n = 2k + 2)$$
$$n - 3 \ (k \geq 3,\ n \neq 2k + 2)$$

Relational complexity

- Sym_n: $\lfloor \log_2 k \rfloor + 2$
- Alt_n: stabilizer height of Sym_n.

Relational spectra: Sym_n: Interval $[2, \text{r.c.}]$
Alt_n: $[2, \text{r.c.}_{\mathrm{Sym}}] \cup [h_{\mathrm{Sym,min}}, h_{\mathrm{Sym,max}}]$ (?)
E.g. Alt_{11}, $k = 3$: $[2, 3] \cup [5, 8]$

Partitions

• The relational complexity of Sym_{2n} on shape $n \times 2$ is n. The relational complexity of Alt_{2n} on shape $n \times 2$ is similar: $n-1$, n, or (for $n=3$) $n+1$

Problem

Estimate the relational complexity of Sym_{nk} on shape $n \times k$, in particular shape $2 \times k$.

The "geometric height" (breadth of the lattice generated by the partitions of shape $n \times k$) is known.

Problem

Is the stabilizer height of Sym_{nk} on $n \times k$ typically equal to this "geometric height?"

Binary almost simple actions

Conjecture

A binary primitive action with simple socle is Sym_n *acting naturally.*

Alternating socle handled by Gill and Spiga.
Eliminate the rest?

• Case study: Socle = Monster (Dalla Volta, Gill, Spiga) *(my understanding of the analysis)*

44 known maximal subgroups up to conjugacy; any others have socle $L_2(13)$ or $L_2(16)$.

Elementary abelian groups

Fact (Gill et al.)

If a binary group contains an elementary abelian p-subgroup $A = \langle a, b \rangle$ of rank 2 with a, b, ab all conjugate, and a fixes a point α, then the order of G_α is divisible by p^2.

Removes 28 of the 44 maximal subgroups, and the two doubtful socles. (Primes 5, 7, 11.)

Suborbit divisors

Remark

Let G be a binary group on Ω.
Suppose that (almost) every transitive binary action of the point stabilizer has degree divisible by the prime p. Then p divides $|\Omega| - 1$.

With a great deal of computation (Spiga) inside the point stabilizers, this eliminates 13 more, leaving:

#9.	$S_3 \times Th$
#15.	$3^{3+2+6+6} : (L_3(3) \times SD_{16})$
#19.	$5^{3+3}.(2 \times L_3(5))$

The holdouts

#9.	$S_3 \times Th$
#15.	$3^{3+2+6+6} : (L_3(3) \times SD_{16})$
#19.	$5^{3+3}.(2 \times L_3(5))$

Examine suborbits, using structural information rather than computation

E.g. $M = S_3 \times Th$:

$$M = N(\langle a \rangle) \text{ (order 3)}$$
$$N_{Th}(b) = (3 \times G_2(3)) : 2 \text{ (maximal, } |b| = 3,\ b \in Th)$$
$$a^g = ab$$
$$M \cap M^g = N_M(ab) = \langle a \rangle \times N_{Th}(b)$$

Orbit of M on Mg permutation isomorphic to primitive action of Th; contradiction.

Some Group Theoretic Questions

Problem

Is there an absolute bound to the minimum relational complexity of some primitive action of every almost simple group?

Problem

Good bounds—especially, lower bounds—for the action of a primitive solvable group.

Problem

Minimal invariant languages for sporadic groups. Interpretation?

Some Combinatorial Questions

Problem

Estimate the complexity of Sym_{nk} (and Alt_{nk}) on partitions of shape $n \times k$. $\rho \approx c_k n$? $c_k << k$?

Problem

Product action: $[{n \atop k}]^d$ under $\mathrm{Sym}_n \wr \mathrm{Sym}_d$

Problem

Aschbacher classes $\mathcal{C}_1$–$\mathcal{C}_8$ (minimum over the cohort).

Problem

When does $\mathrm{A\Gamma L}_1(q)$ have relational complexity 3? 4?

Cohorts

Problem

Complexity spectrum for product actions with socle

$$\mathrm{Alt}_n^d \text{ on } [n]^d$$

Example (Product: Socle Alt_{10}^2 on $[10] \times [10]$)

Group	*Spectrum*
$\mathrm{Sym}_{10} \wr \mathrm{Sym}_2$	$\{2, 4\}$
$\mathrm{Alt}_{10} \wr \mathrm{Sym}_2$	$\{2, 4, 9\}$
$\mathrm{Alt}_{10}^2 .2^2$	$\{2, 4, 9, 10, \ldots, 16\}$
$\mathrm{Alt}_{10}^2 .4$	$\{2, 4, 10, 11, \ldots, 16\}$

Problem

Spectrum for Alt_n *on* k*-sets:* $[2, \rho^+] \cup [h_{\min}, h_{\max}]$*?*