The Galois Group of the Category of Mixed Hodge-Tate Structures

Guangyu Zhu
(joint with Alexander Goncharov)

Yale University

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Goal

Let MHTS (resp. VMHTS_X) be the category of (variations of) $\mathbb{Q}$-mixed Hodge-Tate structures (over X).

1. Construct a Hopf algebra $\mathcal{A}_\bullet = \mathcal{A}_\bullet(\mathbb{C})$;
2. Show that $\mathcal{A}_\bullet$ is the fundamental Hopf algebra of MHTS:

$$\mathsf{MHTS} \sim \mathsf{gComod}(\mathcal{A}_\bullet(\mathbb{C}));$$

3. Generalization: construct a Hopf algebra $\mathcal{A}_\bullet(R)$ for "any" R;
4. Applications, e.g.,

$$\mathsf{VMHTS}_X \sim \mathsf{gComod}_X(H^0(\mathcal{A}_\bullet(\Omega^\bullet_X), \mathcal{D})).$$

Overview

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Notations and conventions

- All (mixed) Hodge structures are over $\mathbb{Q}$.
- All tensor products are over $\mathbb{Q}$, i.e. $\otimes := \otimes_{\mathbb{Q}}$, except that in the generalized case, it is over the field k.
- $\mathbb{C}^* := \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \underset{\exp}{\overset{\sim}{\leftarrow}} \mathbb{C}/\mathbb{Q}(1)$ is a $\mathbb{Q}$-vector space.
  We usually use X or $\exp x$ or $\overline{x}$ to denote an element. ($\overline{x}$ means $x \mod 2\pi i$.)
- Let V be a vector space. Then $T(V) := \bigoplus_{n \geq 0} V^{\otimes n}$ is the tensor algebra.

The Period Matrix of a MHTS

We can use the period matrix to present a mixed Hodge-Tate structure. For example, suppose we have a period matrix

$$\begin{pmatrix} 1 & & \\ -Li_1(z) & 2\pi i & \\ -Li_2(z) & 2\pi i \log z & (2\pi i)^2 \end{pmatrix}$$

Let C_i $(i = 0, 1, 2)$ be the columns and e_i $(i = 0, 1, 2)$ be the standard basis in column space $\mathbb{C}^{(n)}$. Then the matrix correspond to the following mixed Hodge-Tate structure:

$$V_{\mathbb{Q}} = \bigoplus \mathbb{Q} C_i;$$

$$W_0 V = V; W_{-1} V = W_{-2} V = \mathbb{Q} C_1 \oplus \mathbb{Q} C_2; W_{-3} V = W_{-4} V = \mathbb{Q} C_2; W_{-5} V = 0;$$

$$V_{\mathbb{C}} = \bigoplus \mathbb{C} C_i = \bigoplus \mathbb{C} e_i;$$

$$F_1 V = 0; F_0 V = \mathbb{C} e_0; F_{-1} V = \mathbb{C} e_0 \oplus \mathbb{C} e_1; F_{-2} V = V.$$

Single Valued Polylogarithms

Polylogarithms are defined recursively:

$$\mathrm{Li}_1(z) = -\log(1-z); \quad \mathrm{Li}_{n+1}(z) = \int_0^z \mathrm{Li}_n(t)\frac{\mathrm{d}t}{t}.$$

There are some well-known single valued version of them. For example, $\log(z) \rightsquigarrow \log|z|$; $Li_2(z) \rightsquigarrow D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) - \log|z|\,\mathrm{Li}_1(z))$ (Bloch-Wigner). Generalizing such results, we can observe the following single valued functions (that take values in $\mathcal{A}_\bullet$) related to polylogarithms:

Examples

$$\mathrm{Li}_2(z) \rightsquigarrow -\exp(\frac{-\mathrm{Li}_2(z)}{2\pi i}) \otimes 2\pi i + \exp(\log z) \otimes (-\mathrm{Li}_1(z)) \in \mathbb{C}^* \otimes \mathbb{C}.$$

$$\log X \otimes_{\mathbb{Q}} \log Y \rightsquigarrow X \otimes \log Y + Y \otimes \log X - \exp(\frac{\log X \log Y}{2\pi i}) \otimes 2\pi i \in \mathbb{C}^* \otimes \mathbb{C}.$$

Remark: One may as well recover the single valued, real valued function from the construction above. For example, apply $(\mathrm{Re} \circ \log) \otimes \mathrm{Im}$ to the function related to Li_2 in the example recovers Bloch-Wigner dilogarithm.

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A Glance at $\mathcal{A}_\bullet$

Input: $\mathbb{C}$ as a $\mathbb{Q}$-algebra, with a subspace $\mathbb{Q}(1) \subset \mathbb{C}$.
Output: $\mathcal{A}_\bullet(\mathbb{C})$, a graded, commutative Hopf algebra.

The Hopf algebra $\mathcal{A}_\bullet$

$$\mathcal{A}_0 = \mathbb{Q}; \quad \mathcal{A}_1 = \mathbb{C}^*; \quad \mathcal{A}_2 = \mathbb{C}^* \otimes \mathbb{C}.$$

Coproduct:

$$\mathcal{A}_2 \to \mathcal{A}_1 \otimes \mathcal{A}_1$$

$$\mathbb{C}^* \otimes \mathbb{C} \to \mathbb{C}^* \otimes \mathbb{C}^*$$

$$X \otimes y \mapsto X \otimes \exp(y).$$

Product:

$$\mathcal{A}_1 \otimes \mathcal{A}_1 \to \mathcal{A}_2$$

$$\mathbb{C}^* \otimes \mathbb{C}^* \to \mathbb{C}^* \otimes \mathbb{C}$$

$$X \otimes Y \mapsto X \otimes \log Y + Y \otimes \log X - \exp\left(\frac{\log X \log Y}{2\pi i}\right) \otimes 2\pi i.$$

(We already know a quotient $\mathcal{A}_\bullet / \mathcal{A}_{>2}$.)

Recall: Tannakian Formalism

Consider the category MHTS with fiber functor

$$\omega : \ \mathsf{MHTS} \to \mathsf{Vect}\,.$$

Then the Galois group of MHTS is $G_{\mathsf{MHTS}} := \mathsf{Aut}^{\otimes}(\omega)$, i.e.,

$$\mathsf{MHTS} \sim \mathsf{Rep}(G_{\mathsf{MHTS}}) = \mathsf{Rep}(\mathsf{Aut}^{\otimes}(\omega)).$$

The embedding gVect $\sim$ HTS $\to$ MHTS gives a decomposition

$$0 \to U \to G_{\mathsf{MHTS}} \to \mathbb{G}_m \to 0,$$

where U is pro-unipotent.
The forgetful functor $\mathsf{gr}^W : \ \mathsf{MHTS} \to \mathsf{gVect}$ gives a splitting of the short exact sequence, so

$$\mathsf{MHTS} \sim \mathsf{gComod}(\mathcal{O}(U)).$$

Then the theorem says $\mathcal{A}_\bullet(\mathbb{C}) \cong \mathcal{O}(U)$ (canonically).

The Fundamental Hopf Algebra of MHTS

Theorem

There is a canonical equivalence of categories:

$$\mathsf{MHTS} \sim \mathrm{gComod}(\mathcal{A}_\bullet(\mathbb{C})).$$

Sketch of proof.

Let $\mathcal{H}_\bullet$ be the graded Hopf algebra of equivalence classes of framed mixed Hodge-Tate structures. Then

$$\mathsf{MHTS} \sim \mathrm{gComod}(\mathcal{H}_\bullet).$$

There exists a canonical isomorphism, the period map

$$\mathcal{P} : \mathcal{H}_\bullet \to \mathcal{A}_\bullet(\mathbb{C})$$

$$\mathcal{P}(f^n, H, v_0; s) := \sum_{\substack{1 \le k \le n \\ 0 = i_0 < i_1 < \cdots < i_k = n}} (-1)^k \bigotimes_{l=1}^{k} \left([f^{i_l} \mid H, s \mid v_{i_{l-1}}] \otimes (2\pi i)^{\otimes (i_l - i_{l-1} - 1)} \right).$$

□

The Period Map $\mathcal{P}$

Example

$\mathcal{P}$ sends a (split) framed mixed Hodge-Tate structure to an element in $\mathcal{A}_\bullet(\mathbb{C})$. For example, fix $z \in \mathbb{P}^1 - \{0, 1, \infty\}$,

$$\begin{pmatrix} 1 & & \\ -\operatorname{Li}_1(z) & 2\pi i & \\ -\operatorname{Li}_2(z) & 2\pi i \log z & (2\pi i)^2 \end{pmatrix} \mapsto -\exp(\frac{-\operatorname{Li}_2(z)}{2\pi i}) \otimes 2\pi i$$

$$+ \exp(\log z) \otimes (-\operatorname{Li}_1(z)).$$

It is well-defined, for example, the monodromy at 1 gives: $\operatorname{Li}_1(z) \rightsquigarrow \operatorname{Li}_1(z) + 2\pi i$ and $\operatorname{Li}_2(z) \rightsquigarrow \operatorname{Li}_2(z) + 2\pi i \log z$. The right hand side remains the same.

Example: a Graded $\mathcal{A}_\bullet$ Comodule gives a MHTS

There is a explicit way to assign a MHTS to a graded $\mathcal{A}$ comodule. Here is an example:

Example

Let $V_{\mathbb{Q}} = \mathbb{Q}C_0 \oplus \mathbb{Q}C_1 \oplus \mathbb{Q}C_2$, where C_n has weight $-2n$. The $\mathcal{A}$-comodule structure on V, i.e. the map $V \otimes V^\vee \to \mathcal{A}_\bullet$ is given by: (fix $z \in \mathbb{P}^1 - \{0, 1, \infty\}$)

$$C_0 \otimes C_1^\vee \mapsto \overline{-Li_1(z)}; \quad C_1 \otimes C_2^\vee \mapsto \overline{\log z};$$

$$C_0 \otimes C_2^\vee \mapsto -(\overline{-\operatorname{Li}_2(z)/(2\pi i)}) \otimes 2\pi i + \overline{\log z} \otimes (-\operatorname{Li}_1(z)).$$

There exists a lifting to $V \otimes V^\vee \to T(\mathbb{C})$ as a $T(\mathbb{C})$-comodule. Equivalently, we choose a compatible set of values for each multi-valued function above. Then from the expression we can read the period matrix

$$\begin{pmatrix} 1 & & \\ -\operatorname{Li}_1(z) & 2\pi i & \\ -\operatorname{Li}_2(z) & 2\pi i \log z & (2\pi i)^2 \end{pmatrix}$$

which determines a mixed Hodge-Tate structure. Such a construction does not depend on any choice we made above.

Compare: a Non-canonical Description

By Beilinson's vanishing theorem, MHTS has homological dimension 1. Then we know

$$\mathcal{O}(U) \cong T(\bigoplus_{n=1}^{\infty} \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n))).$$

However this isomorphism is not canonical, i.e., it does not work in families.

Example

Weight 2: $\mathbb{C}^*(1) \oplus \mathbb{C}^* \otimes \mathbb{C}^* \cong \mathbb{C}^* \otimes \mathbb{C}$ non-canonically.
(Compare: $0 \to \mathbb{Q}(1) \to \mathbb{C} \to \mathbb{C}^* \to 0$ does not split canonically.)

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Generalization to $\mathcal{A}_\bullet(R)$

We can generalize the construction of $\mathcal{A}_\bullet$ by replacing

- $\mathbb{Q} \rightsquigarrow k$ any field;
- $\mathbb{C} \rightsquigarrow (R, \mathrm{d})$ any dg-algebra over k;
- with $\mathbb{Q}(1) \rightsquigarrow k(1) \subset R$ "Tate line".

Then we get a graded Hopf dg-algebra $(\mathcal{A}_\bullet(R), \mathcal{D})$.

A glance at $\mathcal{A}_\bullet(R)$

$$\mathcal{A}_0 = k; \quad \mathcal{A}_1 = R^0/k(1); \quad \mathcal{A}_2 = R^0/k(1) \otimes R^0 \oplus R^1.$$

The only nontrivial part of $\mathcal{D}$ on $\mathcal{A}_2$ is

$$R^0/k(1) \otimes R^0 \to R^1$$

$$\bar{a} \otimes b \mapsto \mathrm{d}a \cdot b$$

Application: $R = \Omega_X^\bullet$

Let X be a complex manifold. Let $(R, \mathrm{d}) := (\Omega_X^\bullet, \mathrm{d})$ be the de Rham complex of sheaves on X. Then we have a graded Hopf dg-algebra $\mathcal{A}_\bullet(\Omega_X^\bullet)$.
(For regular variety X with normal crossing divisor at infinity, use $\Omega_{\log}$ instead.)

Theorem

the fundamental Hopf algebra of VMHTS *is*

$$H^0(\mathcal{A}_\bullet(\Omega_X^\bullet), \mathcal{D}) = \{x \in \mathcal{A}_\bullet(\mathcal{O}_X) \mid \mathcal{D}(x) = 0\}.$$

In other words,

$$\mathrm{VMHTS} \sim \mathrm{gComod}(H^0(\mathcal{A}_\bullet(\Omega_X^\bullet), \mathcal{D})).$$

Let $\mathrm{Cobar}_\bullet(\mathcal{A}_\bullet(\Omega))$ be the cobar complex of the Hopf algebra. Then

Theorem

The weight n part of the cobar complex $\mathrm{Cobar}_n(\mathcal{A}_\bullet(\Omega))$ *is quasi-isomorphic to the weight n Deligne complex* $\mathbb{Q}_{\mathcal{D}}^\bullet(n)$.

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The Construction of $\mathcal{A}_\bullet(R)$

Given (R, d) a dg-algebra with a Tate line $k(1) \subset R^0$ and a non-zero $t \in k(1)$, we can construct a graded Hopf dg-algebra $\mathcal{A}_\bullet(R)$ (independent of choice of t).
As a vector space $\mathcal{A}_\bullet(R) := T(R)/(k(1) \otimes T(R))$, i.e.

$$\mathcal{A}_\bullet(R) = k \oplus \bigoplus_{n=1}^{\infty} (R/k(1)) \otimes R^{\otimes(n-1)}.$$

The quasi-shuffle product on $T(R)$ can be defined recursively by

$$\begin{aligned}
&(x_1 \otimes x_2 \otimes \cdots \otimes x_p) * (y_1 \otimes y_2 \otimes \cdots \otimes y_q) \\
&= x_1 \otimes \big((x_2 \otimes \cdots \otimes x_p) * (y_1 \otimes y_2 \otimes \cdots \otimes y_q)\big) \\
&+ (-1)^{\deg y_1 (\sum_{i=1}^{p} \deg x_i)} y_1 \otimes \big((x_1 \otimes x_2 \otimes \cdots \otimes x_p) * (y_2 \otimes \cdots \otimes y_q)\big) \\
&- (-1)^{\deg y_1 (\sum_{i=2}^{p} \deg x_i)} x_1 y_1 / t \otimes t \otimes \big((x_2 \otimes \cdots \otimes x_p) * (y_2 \otimes \cdots \otimes y_q)\big)
\end{aligned}$$

The deconcatenation coproduct on $T(R)$ is

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

The Construction of $\mathcal{A}_\bullet(R)$

The differential $\mathcal{D}$ on $T(R)$ is defined recursively by

$$\mathcal{D}(x_1 \otimes \cdots x_n) = \mathrm{d}x_1 \cdot x_2 \otimes \cdots \otimes x_n + (-1)^{\deg x_1} x_1 \otimes \mathcal{D}(x_2 \otimes \cdots \otimes x_n).$$

The cohomological grading of $T(R)$ is given by

$$|x_1 \otimes \cdots \otimes x_n| := \sum \deg x_i.$$

The weight grading of $T(V)$ is given by

$$w(x_1 \otimes \cdots \otimes x_n) := n + \sum \deg x_i.$$

Therefore the product and coproduct respect both gradings, and

$$|\mathcal{D}| = 1, \quad w(D) = 0.$$

Note that such product and coproduct do not make $T(R)$ a bialgebra. However the induced structures on $\mathcal{A}_\bullet(R)$ make it a graded Hopf dg-algebra.

Thank you!