"Cosmic Galois Theory" and Amplitudes in Planar N=4 Super-Yang-Mills Theory


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Rare case where same function space works (conf.)

The suitably normalized amplitude for scattering 6 gluons in planar N=4 SYM in D=4-2ε

\( A_6(\varepsilon; k_2, k_1 + 1) = \frac{\text{BDS-like}(\varepsilon)}{E(u,v,w)} \)

\( u_i = \frac{(k_i + k_{i+1})^2 (k_i + k_{i+1})^2}{(k_i + k_{i+1})^2 (k_i + k_{i+1})^2} \)

\( k_i^2 = 0 \)

\( i = 1, 2, \ldots, 6 \)

depends only on \( u,v,w \)

Conjecture: \( E(u,v,w) \)

expanded perturbatively, \( E = \sum_{L=0}^{\infty} \frac{1}{(8\pi)^L} E^{(L)}(\varepsilon) \)

has coefficients \( E^{(L)}(\varepsilon) \)

which are weight 2L multiple polylogarithms, belonging to a very small subspace \( \mathcal{H} \subseteq \mathcal{Y} \)

\( \mathcal{Y} = \sum_{i=1}^{\infty} \mathcal{Y}(w, y_i) \)

\( \mathcal{Y}(w, y_i) = \int_0^1 \frac{dt}{t-a} \mathcal{Y}(a, \ldots, a; t) \)

\( \mathcal{Y}(a, \ldots, a; t) = \frac{a^p}{p!} \)

\( \mathcal{Y}(0, \ldots, 0) = \frac{1}{y_1 y_2 \ldots} \)
$2 \leq D_6$ dihedral symmetry generated by $g: Y_u \rightarrow Y_u, Y_v \rightarrow Y_v, Y_w \rightarrow Y_w, u \rightarrow v, v \rightarrow w, w \rightarrow u$.

Here the $Y_u, Y_v, Y_w$ are coordinates on $\text{Conf}_6(\mathbb{R}^3) \cong \text{Gr}(4,6)$.

and

$u = \frac{Y_u (1-Y_v)(1-Y_w)}{(1-Y_u Y_v)(1-Y_u Y_w)}$, $v = \frac{Y_v (1-Y_w)(1-Y_u)}{(1-Y_u Y_w)(1-Y_v Y_u)}$, $w = \ldots$

$1-u = \frac{(1-Y_u)(1-Y_u Y_v Y_w)}{(1-Y_u Y_v)(1-Y_u Y_w)}$.

$\Delta_{n-1}[F] = \sum_{s \in S} F^{s_1} \otimes d\nu_{s_1}$

derivatives wrt. coords

$\implies \frac{\partial F}{\partial x_1} = \sum_{s \in S} \frac{F^{s_1} \otimes d\nu_{s_1}}{\partial x^1}$.

$\text{For } Y, \ S_y = S_y' = 3 Y_u, Y_v, Y_w, Y_u Y_v, Y_u Y_w, Y_v Y_w$.

10 letter alphabet:

$1 - Y_u Y_v, 1 - Y_w Y_v, 1 - Y_u Y_w, 1 - Y_u Y_w Y_v$.

However, we want only a 9 letter alphabet:

$S_u = \{ Y_u, Y_v, Y_w, 1-Y_u, 1-Y_v, 1-Y_w, Y_u Y_v, Y_u Y_w, Y_v Y_w \} \subset S_y$.

Parity: $P = g_3: u \rightarrow v, v \rightarrow w, u \rightarrow w, w \rightarrow u, Y_u \rightarrow Y_v, Y_v \rightarrow Y_w$.

$\frac{\partial F}{\partial u, v} = \frac{E_u - E_{1-u}}{u - 1}$

$+ \frac{1}{(1 - u - v) F_{u, v}} + \frac{1}{(1 - v - w) F_{v, w}} + \frac{1}{(1 - w - u) F_{w, u}}$.

Branch cuts in scattering amplitudes

only for $\sum (k_i + k_{i+1})^2 = 0$

$\sum (k_i + k_{i+1} + k_{i+2})^2 = 0$

$\Rightarrow$ at $u, v, w \approx 0$ or $u, v, w \approx \infty$

Not at $u, v, w \approx 1$, not in $y, u, v, w$.

$\Rightarrow$ weight 1 functions in $\mathcal{H}^{(1)} = \mathbb{C} u, v, w, u v w = \{ k u w \}$

$\ln(1 + u) \ln v_i$

Now we impose the action principle for $\mathfrak{g}_{n-1,1}$:

$\Delta_{n-1,1} \mathcal{H} = \mathcal{H} \otimes \text{dlass}$

$i.e.$

$\mathcal{H}^{(n)} = \frac{1}{N!} \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} H_i \text{dlass}_j$

Integrability $\mathcal{H}^{(2)} = 0$

$H_{ij}^{(n)} \leq \mathcal{H}^{(1)}$

$N_i = 3$

But we also need to maintain absence of branch cuts

$\Rightarrow$ $\mathcal{H} = \mathcal{H}^{(1)} + \text{const}$

$\Rightarrow$ constrain constants (MEV's)

$\ln H^{(n-1)}$

And Steinmann relations: no double discontinuities in overlapping channels, $\text{Disc}(k_i + k_{i+1}) \text{Disc}(k_i + k_{i+1} + k_2) A_6 = 0$
Steinmann relations forbid $\ln^2 u$

\[ \equiv \ln(k_1+k_2+k_3)^2 \cdot \ln(k_2+k_4+k_5)^2 \]

allow $\ln^2 a$ where $a = \frac{u}{vw} = \left[ \frac{(k_2+k_3+k_4)^2}{2 \text{ particle invariants}} \right]^{1/2}$

\[ \ln^2 b \]

\[ b = \frac{v}{wu} = \left[ (k_3+k_4+k_5)^2 \right]^{1/2} \]

\[ \ln^2 c \]

\[ c = \frac{w}{uv} = \left[ (k_1+k_2+k_3)^2 \right]^{1/2} \]

\[ \ln a \ln b \]

\[ H^{(2)} = \sum \ln(1-u_i) + \ln^2 \left( \frac{u_i}{u_i - 1 + (u_i - 1)^2} \right) + 4s_2 \]

\[ E^{(1)} = \frac{3}{2} \sum \ln^2 (1-u_i) \leq \frac{2}{9} \]

Also 3 "NMHV" amplitude $E^{(1)} = -\frac{1}{4} \left( \ln^2 b + 4s_2 \right) \leq \frac{2}{9}$

At weight 3, first parity odd ($y_i$-containing) function appears, $\Phi_6 \propto \Delta_{26} \leftrightarrow (S_3 \text{ symmetric})$

Constant term in $\Phi_6$ fixed by branch cut condition

\[ \Phi_6^{yu} = \sum_{i=1}^{3} \ln^2 (1-u_i) + \frac{1}{4} \left[ (\ln^2 b + 4s_2) + (\ln^2 c + 4s_2) \right] \]

Should $\to 0$ as $(u,v,w) \to (0,1,0)$ or $(0,0,1)$ (fixes $S_3$ well.)
At all lower weights $w_t$, $r_t$ is only 3.69 < 372(1)

Just as well, $\frac{\alpha}{\omega}$ is in here.

Why are only free constants

$320 - 320 = 0$

$\alpha = 0.187$

$N_0 = 1.09$ + 1

Lamda spacing

$170 = 169 + 1$

...in a subspace...
in part of co-action principle is built in

What about other components?

\[ \Delta_{p,q} \Psi = \Psi_p \otimes K_q \]

We don't understand \( K_q \) very well for generic \( u,v,w \).

But maybe we can gather evidence for (4) by looking at MZV's and Euler sums at specific points.

Our favorite point is \( (u,v,w) = (1,1,1) \).

Non-singular, dihedrally symmetric, \( \exists 1 \) MZV's

<table>
<thead>
<tr>
<th>( w )</th>
<th>MZV's at ( (1,1,1) ) in ( q'H )</th>
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<tr>
<td>10</td>
<td>( 8_{10}, 4825, 52 + 255_i^2, 257 + 285_i^5 + 115_i^2 - 4525, 5^5 - 655_i^2 )</td>
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<td>9</td>
<td>( 685 - 759, 5257 - 556, 3555 - 559 )</td>
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<td>8</td>
<td>( 66 + 5555 - 555 - 525, 55 )</td>
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<td>7</td>
<td>( 755 - 5255 - 353, 54 )</td>
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<td>6</td>
<td>( 56 )</td>
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<td>5</td>
<td>( 555 - 2525 )</td>
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Many omissions can be explained by the Galois co-action.

E.g.: \( \Delta_{3,3} \Psi_3^2 = 2 \Psi_{3,3} \otimes \Psi_{3,3} \) missing

More generally, act with \( \Theta_{3,3} \) for derivations \( \Theta_{n,n} \) of Brown, 1102.1310
E.g. \( \varphi_{5,3}^m = 0 \)

\[ \varphi_{5,3}^m = -5 \varphi_{3,3}^m \]

\[ \Rightarrow \; 2^3 \left( \varphi_{5,3}^m + 5 \varphi_{3,3}^m - 3 \varphi_{1,3}^m \right) = 5 \varphi_{5,3}^m - 2 \varphi_{2,3}^m \]

\[ \Rightarrow \; 2^5 \left( \varphi_{5,3}^m + 5 \varphi_{3,3}^m - 3 \varphi_{1,3}^m \right) = -5 \varphi_{5,3}^m + 5 \varphi_{3,3}^m = 0 \]

(Euler sums can be obtained at two singular points:

(A) \((u, v, w) \to (\frac{1}{2}, 0, \frac{1}{2}) = 4^k \cdot \text{Euler sums}\)

Get all more Euler sums through weight 5

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Now there is also a \( \frac{1}{2} \)

(B) \((u, v, w) \to (\alpha, 0, \alpha) \)

Dropouts start with \( \ln 2 \) at weight 4.

But one "new" one through weight \( 9! \)

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Final Confession:

To make the coaction principle work, we needed to redefine \( \varepsilon \) (by changing \( A_8 \))

\[ \varepsilon \rightarrow \frac{\varepsilon}{f} \]

where

\[ f = 1 + a^3 (s_2)^2 + a^4 (-10s_5 s_5) \]
\[ + a^5 \left[ \frac{105}{2} s_5 s_7 + \frac{57}{2} (s_5^2) - s_4 (s_3)^2 \right] \]
\[ + a^6 \left[ -294 s_3 s_9 - 651 s_5 s_7 + 7 s_4 (s_3 s_5) + \frac{49}{4} s_8 s_3 \right] \]

\( (a = \frac{1}{8}) \)

What is (the meaning of) \( f \)?

\( f \) is really "mod \( k^{2n} \)" \( n \geq 1 \)

"Cusp anomalous dimension" seems related to \( f \):

\[ \rho - \frac{4a}{f} = -s_2 a + 3s_4 a^2 - \frac{25}{2} s_5 a^3 + \frac{49}{8} s_8 a^4 \]
\[ + \left[ \frac{3 (s_5)^2}{2} + \frac{154 (s_3)^2}{2} - \frac{2745}{8} s_{10} \right] a^5 \]
\[ + \frac{1}{7} a^6 \]