Computing the scale

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Outline of the Talk

Totally disconnected locally compact groups

Definition

A *totally disconnected locally compact (t.d.l.c.)* group is a topological group for which the topology is locally compact, Hausdorff and totally disconnected.

Examples

1. Any G with the discrete topology
2. Profinite groups are compact and totally disconnected

More examples

Examples

3. Lie groups over $\mathbb{Q}_p$ or $F_q((t))$
4. (Completions of) Kac-Moody groups over finite fields
5. Aut(X), X a locally finite graph, *e.g.* $X = T_{s+1}$ or a building
   - Closed subgroups of these automorphism groups
6. AAut(T_{s+1}), groups of almost automorphisms of a tree and their relatives (Neretin's group)

Yet more examples

Examples

7. $G /\!\!/ H$ where $H < G$ is *commensurated*, *i.e.*, $[H : H \cap xHx^{-1}] < \infty$ for every $x \in G$
8. $G < \mathrm{Sym}(\mathbb{N})$, such that $G \curvearrowright \mathbb{N}$ with finite suborbits

This is a list of classes of examples. It is not known how complete this list is.

Problem

Can we do better? Is a classification of t.d.l.c. groups or complete structure theory possible?

Towards a description

Answer

No classification.

A classification of all discrete groups and all profinite groups is too much to ask for.

- Phillip Wesolek defines an *elementary t.d.l.c. group* to be one built from countable discrete and separable profinite groups by a sequence of group theoretic constructions and shows that elementary groups may be treated separately. All solvable groups are elementary.
- 7. and 8. are general constructions. Each may be used to represent essentially all compactly generated t.d.l.c. groups. They are too general to say very much.

Towards a description 2

- 3. – 6. give all known examples of compactly generated, simple groups. Many Lie and Kac-Moody groups are naturally described as groups of commensurators as in 7. Automorphism groups of graphs in 5. are naturally described as permutation groups as in 8.

P.-E. Caprace suggests to focus on closed subgroups of $\mathrm{Aut}(T_{s+1})$ as a 'microcosm' of the class of t.d.l.c. groups. For each s, these are the Chabauty space of $\mathrm{Aut}(T_{s+1})$, which is a compact set.

Progress towards a description

Decomposition Reduce t.d.l.c. groups to smaller pieces:
simple quotients (P.-E.Caprace & N.Monod);
elementary groups (P.Wesolek);
chief series (C.Reid & P.Wesolek)

Local Structure Relate the structure of simple G to properties of compact open subgroups: h.j.i.; decomposable
(Y.Barnea, M.Ershov & T.Weigel; P.-E.Caprace, C.Reid & W.)

Geometry Actions on buildings, locally finite graphs and Cayley-Abels graphs
(R.Möller & B.Krön; B.Rémy, U.Baumgartner & J.Ramagge)

Scale methods Invariants and structures for endomorphisms and automorphisms of t.d.l.c. groups

Compact open subgroups

Theorem (van Dantzig, 1930's)

- *Let G be a t.d.l.c. group and $\mathcal{N}$ be a neighbourhood of the identity. Then there is a compact open subgroup $U \subset \mathcal{N}$.*
- *Every compact t.d.l.c. group is profinite.*

Definition

Subgroups $H, L \leq G$ are *commensurable* if $[H : H \cap L]$ and $[L : H \cap L]$ are both finite.

Any two compact open subgroups of G are commensurable.

The scale and minimizing subgroups

A t.d.l.c. group G for which there is a compact open $U \triangleleft G$ is elementary as defined by P. Wesolek. The next definition applies to inner automorphisms in particular and thus gauges when G is not elementary.

Definition

Let $\alpha \in \mathrm{End}(G)$. The *scale of α* is the positive integer

$$s(\alpha) := \min\left\{[\alpha(U) : \alpha(U) \cap U] : U \overset{\text{compact}}{\leq_O} G\right\}.$$

The compact open subgroup U of G is *minimizing for α* if the minimum is attained at U.

The structure of minimizing subgroups for automorphisms

Theorem

Let $\alpha \in Aut(G)$ and $U \leq G$ be compact and open. Define

$$U_+ := \bigcap_{k\geq 0} \alpha^k(U) \text{ and } U_- := \bigcap_{k\geq 0} \alpha^{-k}(U).$$

Then U is minimizing for α if and only if

TA $U = U_+U_-$; and

TB $U_{++} := \bigcup_{k\geq 0} \alpha^k(U_+)$ is closed.

If U is minimizing, then $s(\alpha) = [\alpha(U_+) : U_+]$.

A similar characterisation holds of subgroups minimising for an endomorphism.

The tidying procedure

Definition

A compact open $U \leq G$ satisfying **TA** and **TB** is *tidy for α*.

The proof of the theorem involves the following steps, which take an arbitrary compact open $U \leq G$ and modify it to be tidy.

Step 1: For the given U, find $N \geq 0$ such that

$$U^{(N)} := \bigcap_{n=0}^{N} \alpha^n(U) \quad \text{satisfies } \textbf{TA}.$$

It may be shown that such N exists and that

$$[\alpha(U^{(N)}) : \alpha(U^{(N)}) \cap U^{(N)}] \leq [\alpha(U) : \alpha(U) \cap U],$$

with equality if and only if U already satisfies **TA**.

The tidying procedure 2

Step 2: For this $U^{(N)}$, define

$$L = \overline{\{g \in G \mid \alpha^k(g) \in U^{(N)} \text{ for almost all } k \in \mathbb{Z}\}}.$$

Then L is a compact, α-invariant subgroup of G.

Step 3: Put $U' = \{u \in U^{(N)} \mid uL \subset LU^{(N)}\}$ and $W := U'L$.

Then W is a compact open subgroup of G and is tidy for α. Further,

$$[\alpha(W) : \alpha(W) \cap W] \leq [\alpha(U^{(N)}) : \alpha(U^{(N)}) \cap U^{(N)}],$$

with equality if and only if $U^{(N)}$ already satisfies **TB** (in which case $L \leq U^{(N)} = W$).

The tidying procedure 3

The proof is completed by showing that, if U, V are any two tidy subgroups, then

$$[\alpha(U) : \alpha(U) \cap U] = [\alpha(V) : \alpha(V) \cap V].$$

Problem

Can this procedure be turned into an algorithm for at least some classes of t.d.l.c. groups?

The group $F^{\mathbb{Z}}$ with shift

Let $G = F^{\mathbb{Z}}$ with F finite. Define the *shift*, $\alpha \in \mathrm{Aut}(G)$, by

$$\alpha(f)(n) = f(n+1) \quad (n \in \mathbb{Z},\ f \in G).$$

$\alpha(G) = G$ and so $s(\alpha) = 1$.

Let $U = \{f \in G \mid f(n) = 1_F, n \in \{0, 3\}\}$.

Step 1: Put $U^{(0)} = U$. Then
$U^{(0)} \cap \alpha(U^{(0)}) = \{f \in G \mid f(n) = 1_F, n \in \{-1, 0, 2, 3\}\}$ and
$[\alpha(U^{(0)}) : \alpha(U^{(0)}) \cap U^{(0)}] = |F|^2$.

Put $U^{(1)} = U \cap \alpha(U)$. Then
$U^{(1)} \cap \alpha(U^{(1)}) = \{f \in G \mid f(n) = 1_F, -2 \leq n \leq 3\}$ and
$[\alpha(U^{(1)}) : \alpha(U^{(1)}) \cap U^{(1)}] = |F|^2$.

The group $F^{\mathbb{Z}}$ with shift 2

Step 1 ctd: Put $U^{(2)} = U \cap \alpha(U) \cap \alpha^2(U) = U^{(1)} \cap \alpha(U^{(1)})$.
Then $U^{(2)} \cap \alpha(U^{(2)}) = \{f \in G \mid f(n) = 1_F, -3 \leq n \leq 3\}$ and
$[\alpha(U^{(2)}) : \alpha(U^{(2)}) \cap U^{(2)}] = |F|$.
We have that $U_+^{(2)} = \{f \in G \mid f(n) = 1_F, n \leq 3\}$ and
$U_-^{(2)} = \{f \in G \mid f(n) = 1_F, n \geq -2\}$. Hence $U^{(2)}$ satisfies **TA**.

Step 2: For this $U^{(2)}$,

$$L = \overline{\{f \in G \mid f(n) = 1_F \text{ for almost all } n\}} = G.$$

Step 3: Then $W = U^{(2)} L = G$, which is minimising.

The automorphism group of a tree with inner automorphism

Let $G = \mathrm{Sym}(T_{s+1})$. Define $\alpha_x \in \mathrm{Aut}(G)$ by $\alpha_x(y) = xyx^{-1}$. Suppose that x is a *translation* through distance d with *axis* ℓ. Let $v \in V(T_{s+1})$ be at distance k from ℓ and let $U = \mathrm{stab}(v)$.

Step 1: Put $U^{(0)} = U$. Then $U^{(0)} \cap \alpha_x(U^{(0)})$ $= \bigcap_{i=0}^{2k+d}\{\mathrm{stab}(v_i) \mid v_i \text{ on the path from } v \text{ to } x.v\}$ and $[\alpha_x(U^{(0)}) : U^{(0)} \cap \alpha_x(U^{(0)})] = (s+1)s^{2k+d-1}$.

Put $U^{(1)} = U \cap \alpha_x(U)$. Then $U^{(1)} \cap \alpha_x(U^{(1)}) =$ $\bigcap\{\mathrm{stab}(v) \mid v \text{ on the subtree spanned by } v,\ x.v \text{ and } x^2.v\}$ and $[\alpha_x(U^{(1)}) : U^{(1)} \cap \alpha_x(U^{(1)})] = (s-1)s^{k+d-1}$.

We have that $U_+^{(1)} = \bigcap\{\mathrm{stab}(x^n.v) \mid n \geq 0\}$ and $U_+^{(1)} = \bigcap\{\mathrm{stab}(x^n.v) \mid n \leq 1\}$. Hence $U^{(1)} = U_+^{(1)}U_-^{(1)}$ and $U^{(1)}$ satisfies **TA**.

The automorphism group of a tree with inner automorphism 2

Step 2: For this $U^{(1)}$, put
$\mathcal{L} = \{y \in \mathrm{Sym}(T_{s+1}) \mid \exists N \text{ such that } y \in \mathrm{stab}(x^n.v) \text{ if } |n| \geq N\}$ and $L = \overline{\mathcal{L}}$. Then elements of $\mathcal{L}$ fix all vertices on the axis ℓ and all but finitely many vertices $x^n.v$. It may be seen that L is the fixator of $V(\ell)$.

Step 3: Then $W = U^{(1)}\mathcal{L} = \bigcap\{\mathrm{stab}(v_i) \mid k \leq i \leq k+d\}$ is minimising for α_x. Hence

$$s(\alpha_x) = [\alpha_x(W) : \alpha_x(W) \cap W] = s^d.$$

A matrix group with conjugation automorphism

Let $G = SL(2, \mathbb{Q}_p)$. Define $\alpha_x(y) = xyx^{-1}$ with $x = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$. Let $U = SL(2, \mathbb{Z}_p)$.

Step 1: Put $U^{(0)} = U$. Then
$U \cap \alpha_x(U) = \left\{ \begin{bmatrix} a_{11} & pa_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$ and
$[U : U \cap \alpha_x(U)] = p + 1$.

The $p + 1$ coset representatives used to count this index are:

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \; b \in \{1, \ldots, p\}; \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

A matrix group with conjugation automorphism 2

Step 1 ctd: Put $U^{(1)} = U \cap \alpha_x(U)$. Then
$U^{(1)} \cap \alpha_x(U^{(1)}) = \left\{ \begin{bmatrix} a_{11} & p^2 a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$ and
$[U^{(1)} : U^{(1)} \cap \alpha_x(U^{(1)})] = p$.
We have that $U_+^{(1)} = \left\{ \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$ and
$U_-^{(1)} = \left\{ \begin{bmatrix} a_{11} & pa_{12} \\ 0 & a_{22} \end{bmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$. Hence $U^{(1)} = U_+^{(1)} U_-^{(1)}$ and
$U^{(1)}$ satisfies **TA**.

Step 2: For this $U^{(1)}$, $\mathcal{L} = \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$ is closed.

Step 3: Then $W = U^{(1)}\mathcal{L} = U^{(1)}$ and $U^{(1)}$ already satisfies **TB**.
Hence $s(\alpha_x) = p$.

Tidy subgroups for commuting automorphisms

Commuting matrices may be simultaneously triangularized.

Theorem

Let $\mathcal{H}$ be a finitely generated abelian group of automorphisms of the t.d.l.c. group G. Then there is a compact open subgroup U of G that is tidy for every α in $\mathcal{H}$.

The commutator of triangular matrices is unipotent.

Theorem

Let α and β be automorphisms of the t.d.l.c. group G and suppose that there is a compact open subgroup U tidy for every automorphism in $\langle \alpha, \beta \rangle$. Then

$$\alpha\beta\alpha^{-1}\beta^{-1}(U) = U.$$

Tidy subgroups as a canonical form

Definition

1. A subgroup $\mathcal{H} \leq \text{Aut}(G)$ is *flat* if there is $U \overset{\text{compact}}{\leq_O} G$ that is tidy for every $\alpha \in \mathcal{H}$.
2. The *uniscalar* subgroup of $\mathcal{H}$ is

$$\mathcal{H}_1 = \left\{\alpha \in \mathcal{H} \mid s(\alpha) = 1 = s(\alpha^{-1})\right\}$$

$\mathcal{H}_1$ is a subgroup because $\alpha \in \mathcal{H}_1$ if and only if $\alpha(U) = U$ for any, and hence all, subgroups tidy for $\mathcal{H}$.

Tidy subgroups as a canonical form

Theorem

Let $\mathcal{H}$ be a finitely generated flat group of automorphisms of the t.d.l.c. group G and suppose that U is tidy for $\mathcal{H}$. Then $\mathcal{H}_1 \triangleleft \mathcal{H}$ and there is $r \in \mathbb{N}$ such that

$$\mathcal{H}/\mathcal{H}_1 \cong \mathbb{Z}^r.$$

1. *There is $k \in \mathbb{N}$ such that*

$$U = U_0 U_1 \dots U_k,$$

where for every $\alpha \in \mathcal{H}$: $\alpha(U_0) = U_0$ and for every $j \in \{1, 2, \dots, k\}$ either $\alpha(U_j) \leq U_j$ or $\alpha(U_j) \geq U_j$.

2. For each $j \in \{1, 2, \ldots, k\}$ there is a homomorphism $\rho_j : \mathcal{H} \to \mathbb{Z}$ and a positive integer s_j such that
$$[\alpha(U_j) : U_j] = s_j^{\rho_j(\alpha)}.$$
3. For each $j \in \{1, 2, \ldots, k\}$,
$$\widetilde{U}_j := \bigcup_{\alpha \in \mathcal{H}} \alpha(U_j)$$
is a closed subgroup of G.
4. The natural numbers r and k, the homomorphisms $\rho_j : \mathcal{H} \to \mathbb{Z}$ and positive integers s_j are independent of the subgroup U tidy for α.

Tidy subgroups as a canonical form

- The numbers $s_j^{\rho_j(\alpha)}$ are analogues of absolute values of eigenvalues for α.
- The subgroups $\bigcup_{\alpha \in \mathcal{H}} \alpha(U_j)$ are the analogues of common eigenspaces for the automorphisms in $\mathcal{H}$.

Example

$G = SL(n, \mathbb{Q}_p)$, $H = \{$ diagonal matrices in $GL(n, \mathbb{Q}_p)\}$ and $\alpha_h(x) = hxh^{-1}$. Then:

- $r = n - 1$;
- $k = n(n-1)$;
- ρ_j are roots of H; and
- $\widetilde{U}_j$ are root subgroups of G.