Automorphism groups of countable structures

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Overview

THEME: Suppose M is a countable first-order structure with a 'rich' automorphism group $\mathrm{Aut}(M)$. Study $\mathrm{Aut}(M)$ as a group and as a topological group.

Involves a mixture of ideas from model theory, group theory, combinatorics, basic topology and descriptive set theory.

Rich: homogeneous structures such as the random graph or the rational numbers as an ordered set; ω-categorical structures; the free group of rank ω, ...

Contents

- Background
- Homogeneous structures; amalgamation classes
- Fraïssé's theorem and generalisations.
- The topology of the symmetric group
- Automorphism groups
- Baire category arguments
- Recovering the topology from the automorphism group
- Normal subgroup structure
- Actions on compact spaces

1.0 Notation and basics: model theory

- L: first-order language (countable).
- We do not distinguish between an L-structure M and its domain.
- If M is an L-structure then $\mathrm{Aut}(M)$ is the automorphism group of M.

DEFINITION: Say that a countably infinite L-structure M is ω*-categorical* if it is determined up to isomorphism amongst countable L-structures by its theory $Th(M)$.

RYLL-NARDZEWSKI THEOREM For a countably infinite L-structure M TFAE: (1) M is ω-categorical;
(2) $\mathrm{Aut}(M)$ has finitely many orbits on M^n for all $n \in \mathbb{N}$.

- The orbits are $\emptyset$-definable sets
- Say that $G \leq \mathrm{Sym}(X)$ is *oligomorphic* if it has finitely many orbits on X^n $\forall n$.

REMARK: The structure on X with relations the G-orbits on X^n is called the *canonical structure* for G on X. If G is oligomorphic this is an ω-categorical structure.

1.1 Homogeneous structures

DEFINITION: An L-structure M is *homogeneous* if isomorphisms beteween finitely generated substructures extend to automorphisms of M.

That is: if $A_1, A_2 \subseteq M$ are f.g. substructures and $f : A_1 \to A_2$ is an isomorphism, then there exists $g \in \mathrm{Aut}(M)$ such that $g|A_1 = f$.

REMARKS:

1. (Warning) Suppose M is any structure. Let M^+ be the canonical structure for $\mathrm{Aut}(M)$ acting on M. Then M^+ is homogeneous and has automorphism group $\mathrm{Aut}(M)$.
2. If L is a finite relational language, then there are only finitely many isomorphism types of L-structure of any finite size. So if M is a homogeneous L-structure, then $\mathrm{Aut}(M)$ is oligomorphic on M.
3. Let L consist of a single 2-ary relation symbol and consider the L-structure $M = (\mathbb{Q}; \leq)$, the rationals with their usual ordering. This is a homogeneous L-structure (use piecewise linear automorphisms).

Amalgamation classes

DEFINITION: A non-empty class $\mathcal{A}$ of finitely generated L-structures is a (Fraïssé) *amalgamation class* if:

1. (IP) $\mathcal{A}$ is closed under isomorphisms;
2. (Hereditary Property, HP) $\mathcal{A}$ is closed under f.g. substructures;
3. (Joint Embedding Property, JEP) if $A_1, A_2 \in \mathcal{A}$ there is $C \in \mathcal{A}$ and embeddings $f_i : A_i \to C$ ($i = 1, 2$);
4. (Amalgamation Property, AP) if $A_0, A_1, A_2 \in \mathcal{A}$ and $f_i : A_0 \to A_i$ are embeddings, there is $B \in \mathcal{A}$ and embeddings $g_i : A_i \to B$ with $g_1 \circ f_1 = g_2 \circ f_2$.

REMARKS:

1. If $\emptyset \in \mathcal{A}$ then JEP follows from AP.
2. Example: The class $\mathcal{A}$ of all finite graphs is an amalgamation class (where $L = \{R\}$). For AP, regard f_1, f_2 as inclusions and let B be the disjoint union of A_1 and A_2 over A_0 with edges $R^{A_1} \cup R^{A_2}$. Take g_1, g_2 to be the natural inclusions.
   We refer to B as the *free amalgam* of A_1, A_2 over A_0.

Fraïssé's Theorem

DEFINITION Suppose M is an L-structure. The *age* of M, $\mathrm{Age}(M)$ is the class of structures isomorphic to some f.g. substructure of M.

THEOREM: (Fraïssé's Theorem)

1. If M is a countable, homogeneous L-structure, then $\mathrm{Age}(M)$ is an amalgamation class.
2. Conversely, if $\mathcal{A}$ is an amalgamation class of countable L-structures, with countably many isomorphism types, then there is a countable homogeneous L-structure M with $\mathcal{A} = \mathrm{Age}(M)$.
3. Suppose $\mathcal{A}$ is as in (2) and M is a countable homogeneous L-structure with age $\mathcal{A}$. Then M has the property that if $A \subseteq M$ is f.g. and $f : A \to B$ is an embedding with $B \in \mathcal{A}$, then there is an embedding $g : B \to M$ with $g(f(a)) = a$ for all $a \in A$. This property determines M up to isomorphism amongst countable structures with age $\mathcal{A}$.

DEFINITION: The structure M is determined up to isomorphism by $\mathcal{A}$ and is referred to as the *Fraïssé limit*, or *generic structure* of $\mathcal{A}$.

Examples 1

- The class of all finite graphs is an amalgamation class. The Fraïssé limit is the *random graph*.
- If $n \geq 3$, let K_n denote the complete graph on n vertices. The class of all finite graphs which do not embed K_n is an amalgamation class and the Fraïssé limit is sometimes called the generic K_n-free graph.
- (Henson digraphs) A *tournament* is a directed graph with the property that for every two vertices a, b, one of $(a, b), (b, a)$ is a directed edge. If $\mathcal{T}$ is a set of finite tournaments, the class $\mathcal{A}(\mathcal{T})$ of finite directed graphs which do not embed any member of $\mathcal{T}$ is an amalgamation class (free amalgamation). There are $2^{\aleph_0}$ countable homogeneous digraphs which can be constructed in this way.

Examples 2

- The class of all finite linear orders is an amalgamation class (but we cannot use free amalgamation). The Fraïssé limit is isomorphic to $(\mathbb{Q}; \leq)$.
- The class of all finite partial orders is an amalgamation class.
- The class of all finite groups is an amalgamation class. The generic structure is Philip Hall's universal locally finite group.

Sketch proof of Fraïssé's Theorem (2,3)

GIVEN: countable amalgamation class $\mathcal{A}$.

CONSTRUCTION: Build M inductively as the union of a chain of structures in $\mathcal{A}$:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots.$$

When doing this we ensure that:

- if $C \in \mathcal{A}$, then C embeds into some A_i;
- if A is a f.g. substructure of A_i and $f : A \to B \in \mathcal{A}$, then there is $j > i$ such that there is an embedding $g : B \to A_j$ with $g(f(a)) = a$ for all $a \in A$.

Countably many tasks to perform here.

A task of the first form can be performed using JEP.

For the second, suppose the construction has reached stage $k > i$. At the next stage we can take A_{k+1} which solves the amalgamation problem $A \to A_k$ (inclusion), $f : A \to B$. Specifically, using AP we obtain $h : A_k \to A_{k+1}$ (which can be taken as inclusion), and $g : B \to A_{k+1}$ with $g(f(a)) = h(a) = a$ for all $a \in A$, as required.

Homogeneity of M

- Suppose M_1, M_2 are countable and have the property that if $A \subseteq M_i$ is f.g. and $f : A \to B$ is an embedding with $B \in \mathcal{A}$, then there is an embedding $g : B \to M_i$ with $g(f(a)) = a$ for all $a \in A$.
- Suppse A_i is a f.g. substructure of M_i and $h : A_1 \to A_2$ is an isomorphism.
- Use a back-and-forth argument to show that h extends to an isomorphism $g : M_1 \to M_2$.

1.2 An extension

GIVEN:

- Class $\mathcal{K}$ of f.g L-structures
- A distinguished class of f.g. substructures $A \sqsubseteq B$ ('A is a *nice* substructure of B'

If $B \in \mathcal{K}$, an embedding $f : A \to B$ is a $\sqsubseteq$-embedding if $f(A) \sqsubseteq B$.
ASSUME: $\sqsubseteq$ satisfies:

(N1) If $B \in \mathcal{K}$ then $B \sqsubseteq B$ (so isomorphisms are $\sqsubseteq$-embeddings);

(N2) If $A \sqsubseteq B \sqsubseteq C$ (and $A, B, C \in \mathcal{K}$), then $A \sqsubseteq C$ (so if $f : A \to B$ and $g : B \to C$ are $\sqsubseteq$-embeddings, then $g \circ f : A \to C$ is a $\sqsubseteq$-embedding).

(N3) Suppose $A \sqsubseteq B \in \mathcal{K}$ and $A \subseteq C \subseteq B$ with $C \in \mathcal{K}$. Then $A \sqsubseteq C$.

Nice amalgamation classes

Say that $(\mathcal{K}, \sqsubseteq)$ is an *amalgamation class* if:

- $\mathcal{K}$ is closed under isomorphisms and has countably many isomorphism types (and countably many embeddings between any pair of elements);
- $\mathcal{K}$ is closed under $\sqsubseteq$-substructures;
- $\mathcal{K}$ has the JEP for $\sqsubseteq$-embeddings;
- $\mathcal{K}$ has AP for $\sqsubseteq$-embeddings: if A_0, A_1, A_2 are in $\mathcal{K}$ and $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ are $\sqsubseteq$-embeddings, there is $B \in \mathcal{K}$ and $\sqsubseteq$-embeddings $g_i : A_i \to B$ (for $i = 1, 2$) with $g_1 \circ f_1 = g_2 \circ f_2$.

REMARKS:

1. If $\sqsubseteq$ is just 'substructure' this is as before.
2. The notion $A \sqsubseteq B$ is only defined when B is f.g.
   If M is a countable L-structure and there are f.g. $M_i \subseteq M$ (with $i \in \mathbb{N}$) such that $M = \cup_{i \in \mathbb{N}} M_i$ and $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$. Then for f.g. $A \subseteq M$ we define $A \sqsubseteq M$ to mean that $A \sqsubseteq M_i$ for some $i \in \mathbb{N}$. The condition N3 guarantees that this does not depend on the choice of M_i.

The generalization

THEOREM: Suppose $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class of finitely generated L-structures and $\sqsubseteq$ satisfies (N1, N2, N3). Then there is a countable L-structure M and f.g. substructures $M_i \in \mathcal{K}$ (for $i \in \mathbb{N}$) such that:

1. $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$ and $M = \cup_{i \in \mathbb{N}} M_i$;
2. every $A \in \mathcal{K}$ is isomorphic to a $\sqsubseteq$-substructure of M;
3. (Extension Property) if $A \sqsubseteq M$ is f.g. and $f : A \to B \in \mathcal{K}$ is a $\sqsubseteq$-embedding then there is a $\sqsubseteq$-embedding $g : B \to M$ such that $g(f(a))$ for all $a \in A$.

Moreover, M is determined up to isomorphism by these properties and if $A_1, A_2 \sqsubseteq M$ are f.g. and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphsim of M.

The proof is essentially the same as that of Fraïssé's Theorem.

Examples

1. (2-out digraphs) Let $\mathcal{K}$ consist of the set of finite directed graphs where every vertex has at most 2 directed edges coming out of it. For $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ if whenever $a \in A$ and $a \to b$ is a directed edge in B, then $b \in A$. Then $\sqsubseteq$ satisfies N1, N2, N3 and $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class (where the amalgamation is just free amalgamation).
2. (Free groups) Let $\mathcal{K}$ be the class of finitely generated free groups. For f.g. $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean that A is a free factor of B. This clearly satisfies N1, N2 and N3 also holds. Moreover $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class and the generic structure is the free group of rank ω.

The Hrushovski construction

- Suppose A is a finite graph.
- $\delta(A) = 2|A| - |\text{Edges}(A)|$ (Predimension)
- $\mathcal{K} = \{A : \delta(X) \geq 0 \text{ for all } X \subseteq A\}$.
- If $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean $\delta(A) \leq \delta(B')$ whenever $A \subseteq B' \subseteq B$.

This satisfies N1, N2, N3 and $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class (where the amalgamation can be taken as free amalgamation).

REMARKS: (1) The generic structure here is the undirected version of the generic structure in Example 1.
(2) There is a variation on this which can be used to produce ω-categorical structures.

2.1 Notation and Basics: permutation groups

G is a group acting on a set X and $a \in X$.

- the G-orbit which contains a is $\{ga : g \in G\} \subseteq X$.
- If there is a unique G-orbit on X we say that G is transitive on X.
- $G_a = \{g \in G : ga = a\}$ is the stabilizer of a in G.
- There is a canonical bijection, respecting the G-action, betweeen the set of left cosets of G_a in G and the G-orbit containing a, given by
$$gG_a \mapsto ga.$$
- In particular, the index of G_a in G is the cardinality of the G-orbit which contains a. (*Orbit- Stabilizer Theorem*.)
- Also consider G acting on X^n (for $n \in \mathbb{N}$) or the power set $\mathcal{P}(X)$.
- If $A \subseteq X$ the pointwise stabilizer of A in G is $G_{(A)} = \{g \in G : ga = a \; \forall a \in A\}$.

Exercise: If X is countable and A is a finite subset of X, then $G_{(A)}$ is a subgroup of countable index in G.

2.2 The topology of $\mathrm{Sym}(X)$

Regard the symmetric group $\mathrm{Sym}(X)$ as a topological group: open sets are unions of cosets of pointwise stabilizers of finite sets.

If $G \leq \mathrm{Sym}(X)$ we give this the relative topology. So the basic open sets in $G \leq \mathrm{Sym}(X)$ are of the form $gG_{(A)}$ for $A \subseteq_{fin} X$ and $g \in G$. Note here that

$$G_{(A)} = \{h \in G : ha = a \ \forall a \in A\}$$

so

$$gG_{(A)} = \{h \in G : h|A = g|A\}.$$

- Each basic open set is also closed. So G is *totally disconnected*.
- If X is countable, there are countably many of these basic open sets (each is determined by a map between finite subsets of X): so G is *second countable*.
- In particular, if X is countable then G is *separable*.

Closed subgroups

LEMMA: Suppose $G \leq \mathrm{Sym}(X)$. Then the closure of G in $\mathrm{Sym}(X)$ is

$$\bar{G} = \{g \in \mathrm{Sym}(X) : gY = Y \text{ for all } G\text{-orbits } Y \text{ on } X^n, \ \forall n\}.$$

Proof.

- Show that if $Y \subseteq X^n$ then $\{h \in \mathrm{Sym}(X) : hY = Y\}$ is closed.
- So $\{h \in \mathrm{Sym}(X) : hY = Y \text{ for all } G\text{-orbits } Y \text{ on } X^n, \ \forall n\}$ is closed and clearly it contains G. So it contains $\bar{G}$.
- Suppose $h \in \mathrm{Sym}(X)$ preserves the G-orbits on X^n for all n. An open neighbourhood O of h is specified by $h|\bar{y}$ for some finite tuple $\bar{y}$. As $h\bar{y}$ is in the same G-orbit as $\bar{y}$ there is $g \in G$ with $g\bar{y} = h\bar{y}$. Thus $g \in O$. This shows that $h \in \bar{G}$.

□

Closed subgroups (2)

COROLLARY: A subgroup G of $\mathrm{Sym}(X)$ is closed iff G is the automorphism group of some first-order structure on X.

Proof.

A first-order structure on X is specified by relations and functions on X. So the automorphsim group is the intersection of the setwise stabilisers of certain subsets of M^n for various n. This is a closed subgroup.

Conversely, if $G \leq \mathrm{Sym}(X)$ consider the structure on X which has a relation for each G-orbit on X^n, for each finite n. The automorphism group of this structure is $\bar{G}$. So if G is closed, the automorphism group is G. □

EXERCISE: Suppose $G \leq \mathrm{Sym}(X)$. Then G is compact iff G is closed in $\mathrm{Sym}(X)$ and all G-orbits on X are finite.

Metrizability

If X is countable (say $X = \mathbb{N}$), the topology on $\mathrm{Sym}(X)$ is separable and complete metrizable.

Consider d given by, for $g_1 \neq g_2$,

$$d(g_1, g_2) = 1/n \text{ where } n \text{ is as small as possible with } g_1 n \neq g_2 n.$$

This is a metric for the topology, but it is not complete. To obtain a complete metric, consider

$$d'(g_1, g_2) = d(g_1, g_2) + d(g_1^{-1}, g_2^{-1}).$$

This is a complete metric for the topology. So if X is countable, then any closed $G \leq \mathrm{Sym}(X)$ is a *Polish group* (a topological group which is separable and complete metrizable).

2.3 Using the topology

Let $S_\infty = \mathrm{Sym}(\mathbb{N})$. Note that $|S_\infty| = 2^{\aleph_0}$.

THEOREM: Suppose $G \leq S_\infty$ is closed. Then either $|G| = 2^{\aleph_0}$ or there exists a finite $Y \subseteq \mathbb{N}$ with $G_{(Y)} = 1$.

Proof.

Consider the isolated points in G.
As G is a topological group, either all points are isolated or no points are isolated (i.e. G is *perfect*).
In the first case, the identity element is isolated so there is a basic open set contained in $\{1\}$; the only way this can happen is if $G_{(Y)} = 1$ for some finite Y.
In the second case, G is a non-empty perfect complete space, so contains a copy of the Cantor set. In particular $|G| = 2^{\aleph_0}$. □

3.1 The small index property

DEFINITION: A countable structure M (or its automorphism group $\mathrm{Aut}(M)$) has the *small index property* (SIP) if whenever H is a subgroup of $G = \mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$, then H is open.
In other words, if $|G : H| < 2^{\aleph_0}$, then there is a finite $A \subseteq M$ with $H \geq G_{(A)}$.
(Note that the first formulation makes sense in an arbitrary topological group.)
REMARKS:

1. If $H \leq G$ is open then $|G : H| \leq \aleph_0$.
2. The SIP implies that we can recover the topology on G from its group-theoretic structure: the open subgroups are precisely the subgroups of small index and the cosets of these form a base for the topology.

Biinterpretability

For a countable ω-categorical structure M the topological group $\mathrm{Aut}(M)$ determines M *up to biinterpretability*.

We cannot expect to recover M completely.

- Consider M with automorphism group $G = \mathrm{Sym}(M)$.
- This acts on N, the set of subsets of size 2 from M.
- Let $G_1 \leq \mathrm{Sym}(N)$ be the set of permutations induced by by this action.
- G_1 is closed in $\mathrm{Sym}(N)$, so we can regard G_1 as the automorphism group of a structure on N.
- The isomorphism $G \to G_1$ (given by the identity map) is a homeomorphism.

So the structures M and N have isomorphic topological automorphism groups even though they are different structures.

SIP: Examples and History

- SIP proved for $\mathrm{Sym}(\mathbb{N})$ by Dixon, Neumann and Thomas (1986).
- SIP for general linear groups and classical groups over countable fields (Evans, 1986, 1991)
- SIP for $\mathrm{Aut}(\mathbb{Q}; \leq)$ (Truss, 1989).
- Different method introduced by Hodges, Hodkinson, Lascar and Shelah (1993) and used to prove SIP for the random graph.
- Approach extended to general Polish groups by Kechris and Rosendal (2007).

In what follows we will follow the presentation of Kechris and Rosendal. There is also work of M Rubin on recovering an ω-categorical structure from its automorphism group.

A counterexample

An ω-categorical structure without SIP (Cherlin and Hrushovski):

Language L: $2n$-ary relation symbol E_n for each $n \in \mathbb{N}$.
$\mathcal{C}$: the class of finite L-structures A in which E_n is an equivalence relation on n-tuples of distinct elements of A with at most 2 classes.

This is an amalgamation class; call the generic structure M.

For each n there are two equivalence classes of distinct n-tuples from M and every permutation of these equivalence classes extends to an automorphism of M.

So $G = \mathrm{Aut}(M)$ has a closed normal subgroup G^0 consisting of automorphisms which fix all equivalence classes and the quotient group is topologically isomorphic to the direct product C_2^ω (where C_2 is the cyclic group with 2 elements).

Assuming the Axiom of Choice, this has non-open subgroups of index 2.

A question

Is the Cherlin - Hrushovski construction essentially the only obstruction to the SIP for an ω-categorical structure?

DEFINITION: Say that an ω-categorical M is *G-finite* if for every open subgroup $H \leq \mathrm{Aut}(M)$, the intersection of the open subgroups of finite index in H is of finite index in H.

Note that in the example, the intersection of the open subgroups of finite index in G is G^0.

Question

If M is a countable ω-categorical structure which is G-finite, does M have the SIP?

Baire Category

DEFINITIONS: Suppose W is a topological space.

- $Z \subseteq W$ is *nowhere dense* if its closure $\bar{Z}$ contains no non-empty open subset of W. Equivalently, $W \setminus \bar{Z}$ is dense in W.
- $Y \subseteq W$ is *meagre* if it is a countable union of nowhere dense sets.
- $X \subseteq W$ is *comeagre* if its complement is meagre. So this means that X contains the intersection of a countable family of dense open sets.

REMARKS: A countable union of meagre sets is meagre and the meagre subsets of W form a σ-ideal in the algebra of subsets of W; so we may think of them as 'small' subsets of W.

THEOREM: (Baire Category Theorem) Suppose W is a complete metrizable space. Then every comeagre subset of W is dense in W. Equivalently, the intersection of any countable family of dense open subsets of W is dense in W.

A simple application

COROLLARY: Suppose $G \leq S_\infty$ is closed and H is a closed subgroup of G. If $|G : H| \leq \aleph_0$ then H is open in G, that is, $H \geq G_{(A)}$ for some finite set A.

Proof.

Suppose H does not contain $G_{(A)}$ for any finite A.
So the complement of H is dense; therefore it is a dense open set.
The same is true for each coset of H.
If there are only countably many cosets their complements form a countable family of dense open subsets of G with empty intersection.
This contradicts BCT. □

Generic automorphisms

Consider the action of a topological group G on the direct product G^n by conjugation:

$$(g_1, \ldots, g_n) \overset{h}{\mapsto} (hg_1h^{-1}, \ldots, hg_nh^{-1}).$$

If we give G^n the product topology, this is a continuous action, which we refer to as the conjugation action.

DEFINITION: Suppose G is a Polish group. We say that G has *ample homogeneous generics* (ahg's) if for each $n > 0$, there is a comeagre orbit of G on G^n (with the conjugation action).

Theorem

(Kechris - Rosendal; Hodges, Hodkinson, Lascar and Shelah) Suppose G is a Polish group with ample homogeneous generics. Then G has the SIP.

REMARKS: Ahg is a strong property. It does not hold for $\mathrm{Aut}(\mathbb{Q}; \leq)$ (fails for $n = 2$).

Finding ahg's

$(\mathcal{K}, \sqsubseteq)$: an amalgamation class of f.g. L-structures (and N1, N2, N3 hold).
M: generic structure of $(\mathcal{K}, \sqsubseteq)$; $G = \mathrm{Aut}(M)$.
If $A \in \mathcal{K}$, then a *partial automorphism* of A is an isomorphism $f : A_1 \to A_2$ where $A_1, A_2 \sqsubseteq A$.
(Partial automorphisms of M are defined similarly.)

Theorem

Suppose that $(\mathcal{K}, \sqsubseteq)$ has the extension property for partial automorphisms (EPPA) and the amalgamation property for automorphisms (APA).
Then $G = \mathrm{Aut}(M)$ has ample homogeneous generics.

EPPA and APA

DEFINITION: (1) Say that $(\mathcal{K}, \sqsubseteq)$ has EPPA (or the *Hrushovski property*) if whenever $A \in \mathcal{K}$ there is a $A \sqsubseteq B \in \mathcal{K}$ such that every partial automorphism of A extends to an automorphism of B.

(2) Say that $(\mathcal{K}, \sqsubseteq)$ has APA if the following holds. Suppose $B \sqsubseteq C_1, C_2 \in \mathcal{K}$. Then there are $D \in \mathcal{K}$ and $\sqsubseteq$-embeddings $\beta_i : C_i \to D$ (which we will regard as inclusions) with $\beta_1(b) = \beta_2(b)\,\forall b \in B$, such that:

- whenever $h_i \in \mathrm{Aut}(C_i)$ are such that $h_i B = B$ and $h_1|B = h_2|B$, then $h_1 \cup h_2$ extends to an automorphism of D.

We will also require this condition without B (so, the joint embedding property).

Examples 1

Let $\mathcal{K}$ be the class of finite graphs (and $\sqsubseteq$ is just embedding). So the generic structure M is the random graph.

APA: If D is the free amalgmation of C_1 and C_2 over B and $f_i \in \mathrm{Aut}(C_i)$ stabilize B and have the same restriction to B, then their union is an automorphism of D.

EPPA: a theorem of Hrushovski.

The result generalises to other free amalgamation classes.

Examples 2

Let $\mathcal{K}$ be the class of finitely generated free groups and $\sqsubseteq$ dentote being a free factor.

The free product with amalgamation gives APA.

EPPA works without needing to extend A.

So the free group of rank ω has the SIP (Bryant and Evans, 1997).

EPPA+APA implies AHGs (1)

The following general result is from notes of C. Rosendal. It translates into a condition for AHGs known as the *weak amalgamation property*, or the *almost-amalgamation property* (due to A. Ivanov).

Suppose we have a continuous action of a Polish group G on a Polish space X. Write $\subseteq_{op}$ to mean 'is a non-empty open subset of'.

Theorem

The following are equivalent:

1. *There is a comeagre G-orbit on X;*
2. *G is topologically transitive on X and whenever $1 \in U \subseteq_{op} G$ and $V \subseteq_{op} X$, there is $W \subseteq_{op} V$ such that U is topologically transisitve on W.*

U is topologically transitive on W means: whenever $W_1, W_2 \subseteq_{op} W$ there is $g \in U$ with $gW_1 \cap W_2 \neq \emptyset$.

EPPA+APA implies AHGs (2)

SET-UP: $(\mathcal{K}, \sqsubseteq)$: an amalgamation class of f.g. L-structures (and N1, N2, N3 hold).
M: generic structure of $(\mathcal{K}, \sqsubseteq)$; $G = \mathrm{Aut}(M)$.
Assume: $(\mathcal{K}, \sqsubseteq)$ has EPPA and APA.

- Want a comeagre orbit of G on $X = G^n$, so verify the criterion in the previous result.
- Have a base of open subsets of X parametrised by $(A; f_1, \ldots, f_n)$ where $A \sqsubseteq M$ and f_i partial automorphisms of A.
- translate the conditions into a condition on $(\mathcal{K}, \sqsubseteq)$: WAP.

3.3 Normal subgroup structure

Some classical results:

THEOREM (J. SCHREIER AND S. ULAM, 1933) Suppose X is countably infinite. If $g \in \mathrm{Sym}(X)$ moves infinitely many elements of X, then every element of $\mathrm{Sym}(X)$ is a product of conjugates of g. In particular, $\mathrm{Sym}(X)/\mathrm{FSym}(X)$ is a simple group.

THEOREM (A. ROSENBERG, 1958). Suppose V is a vector space of countably infinite dimension over a field K. If $FGL(V)$ denotes the elements of $GL(V)$ which have a fixed point space of finite codimension, then $GL(V)/(K^{\times}.FGL(V))$ is a simple group.

THEOREM (G. HIGMAN, 1954). The non-trivial, proper normal subgroups of $G = \mathrm{Aut}(\mathbb{Q}; \leq)$ are the left-bounded automorphisms, $L = \{g \in G : \exists a\; g|(-\infty, a) = id\}$, the right-bounded automorphisms $R = \{g \in G : \exists a\; g|(a, \infty)\}$ and $B = L \cap R$.

THEOREM (J. TRUSS, 1985). Let Γ be the countable random graph. Then $\mathrm{Aut}(\Gamma)$ is simple.

TEMPTING IDEA: Automorphism groups of ‘nice’ countable structures should not have any non-obvious normal subgroups.

Need for caution:

Example (M. Droste, C. Holland, D. Macpherson)

The automorphism group of a countable, homogeneous semilinear order has $2^{2^{\aleph_0}}$ normal subgroups.

A general result

Theorem (D. Lascar, 1992)

Suppose M is a countable saturated structure with a $\emptyset$-definable strongly minimal set D. Suppose that $M = \mathrm{acl}(D)$. Suppose $g \in G = \mathrm{Aut}(M/\mathrm{acl}(\emptyset))$ is unbounded, i.e. for every $n \in \mathbb{N}$ there is some finite $X \subseteq D$ with $\dim(X \cup gX) \geq n + \dim(X)$. Then G is generated by the conjugates of g.

- Implies the results for $\mathrm{Sym}(X)$ and $GL(V)$.
- Proof uses Polish group arguments.
- Ideas used by T. Gardener (1995) to prove analogue of Rosenberg's result for classical groups over finite fields.
- Used by Z. Ghadernezhad and K. Tent (2012) to prove simplicity of automorphism groups of certain generalized polygons and so obtain new examples of simple groups with a BN-pair.

Recent general results

THEOREM (D. MACPHERSON AND K. TENT, 2011): Suppose M is a countable, transitive homogeneous relational structure whose age has free amalgamation. Suppose $\mathrm{Aut}(M) \neq \mathrm{Sym}(M)$. Then
(a) $\mathrm{Aut}(M)$ is simple;
(b) (Melleray) if $1 \neq g \in \mathrm{Aut}(M)$ then every element of G is a product of 32 conjugates of $g^{\pm 1}$.

NOTE: This implies Truss' result and unpublished results of M. Rubin (1988).

K. Tent and M. Ziegler (2012) generalized this to the case where M has a *stationary independence relation* $\mathop{\smile\!\!\!\!\!\!|}$ and used this to prove:

THEOREM: Suppose U is the Urysohn rational metric space. If $g \in \mathrm{Aut}(U)$ is not bounded, then every automorphism of U is a product of 8 conjugates of g.

Stationary independence relations

NOTATION/ TERMINOLOGY:

- M is a countable first-order structure;
- $G = \mathrm{Aut}(M)$;
- cl is a G invariant, finitary closure operation on subsets of M;
- If $X \subseteq_{fin} M$ and a is fixed by G_X, then $a \in \mathrm{cl}(X)$ (where $G_X = \{g \in G : gx = x \ \forall x \in X\}$).
- $\mathcal{X} = \{\mathrm{cl}(A) : A \subseteq_{fin} M\}$;
- $\mathcal{F}$ consists of all maps $f : X \to Y$ with $X, Y \in \mathcal{X}$ which extend to automorphisms of M. Call these *partial automorphisms*.

EXAMPLE: Take cl to be algebraic closure in M. So, for example, if M is the Fraïssé limit of a free amalgamation class, then $\mathrm{acl}(X) = X$ for all $X \subseteq M$.

In what follows, $\mathop{\smile\hskip -0.9em ^{\vert}}$ is a relation between subsets A, B, C of M: written $A \mathop{\smile\hskip -0.9em ^{\vert}}_B C$ and pronounced 'A is independent from C over B.'

Definition:

We say that $\mathop{\smile\!\!\!\!\!\!|}$ is a *stationary independence relation compatible with* cl if for $A, B, C, D \in \mathcal{X}$ and finite tuples a, b:

1. (Compatibility) We have $a \mathop{\smile\!\!\!\!\!\!|}_b C \Leftrightarrow a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}(b)} C$ and
$$a \mathop{\smile\!\!\!\!\!\!|}_B C \Leftrightarrow e \mathop{\smile\!\!\!\!\!\!|}_B C \text{ for all } e \in \mathrm{cl}(a, B) \Leftrightarrow \mathrm{cl}(a, B) \mathop{\smile\!\!\!\!\!\!|}_B C.$$
2. (Invariance) If $g \in G$ and $A \mathop{\smile\!\!\!\!\!\!|}_B C$, then $gA \mathop{\smile\!\!\!\!\!\!|}_{gB} gC$.
3. (Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}_B C \cup D$, then $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{B \cup C} D$.
4. (Transitivity) If $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{B \cup C} D$, then $A \mathop{\smile\!\!\!\!\!\!|}_B C \cup D$
5. (Symmetry) If $A \mathop{\smile\!\!\!\!\!\!|}_B C$, then $C \mathop{\smile\!\!\!\!\!\!|}_B A$.
6. (Existence) There is $g \in G_B$ with $g(A) \mathop{\smile\!\!\!\!\!\!|}_B C$.
7. (Stationarity) Suppose $A_1, A_2, B, C \in \mathcal{X}$ with $B \subseteq A_i$ and $A_i \mathop{\smile\!\!\!\!\!\!|}_B C$. Suppose $h : A_1 \to A_2$ is the identity on B and $h \in \mathcal{F}$. Then there is some $k \in \mathcal{F}$ which contains $h \cup \mathrm{id}_C$ (where id_C denotes the identity map on C).

Remarks and examples

1. Tent and Ziegler consider this where $\mathrm{acl}(X) = X$ and $\mathrm{cl}(X) = X \ \forall X$.
2. Suppose M is the Fraïssé limit of a free amalgamation class (of relational structures). Let $\mathrm{cl}(X) = X \ \forall X$. Define $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ to mean $A \cap C \subseteq B$ and $A \cup B$, $C \cup B$ are freely amalgamated over B. This is a stationary independence relation on M.
3. Suppose M is a countable-dimensional vector space over a countable field K. So $G = GL(M)$. Let cl be linear closure and take $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ to mean that $\mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B) = \mathrm{cl}(B)$. This gives a stationary independence relation.
4. For all $a \in M$ and finite X we have $a \mathop{\smile\!\!\!\!\!\!\mid}_X \mathrm{cl}(X)$. Moreover $a \mathop{\smile\!\!\!\!\!\!\mid}_X a$ iff $a \in \mathrm{cl}(X)$.

Moving almost maximally

DEFINITION: Say that $g \in G$ moves almost maximally if for all $B \in \mathcal{X}$ and $a \in M$ there is a' in the G_B-orbit of a such that

$$a' \underset{B}{\downarrow} ga'.$$

EXAMPLE 1: Suppose $(M; \mathrm{cl}; \downarrow)$ is the vector space example. If $g \in G$ does not move almost maximally, then for some finite dimensional subspace B, for all $v \in M$ we have $gv \in \langle v, B \rangle$. Thus g acts as a scalar α on M/B. So $(\alpha^{-1}g - 1)v \in B$ for all v and it follows that g is a scalar multiple of a finitary transformation.

EXAMPLE 2: Suppose $(M; \mathrm{cl}; \downarrow)$ is the free amalgamation example. Suppose also that $G = \mathrm{Aut}(M)$ is transitive on M and $G \neq \mathrm{Sym}(M)$. If $1 \neq g \in G$, then g moves infinitely many points of each G_B-orbit (for each finite $B \subseteq M$) and using a back-and-forth argument, one shows that there is $h \in G$ such that $[g, h] = g^{-1}h^{-1}gh$ moves almost maximally.

The following is a modification of the result of Tent and Ziegler (due to DE, Z. Ghadernezhad and K. Tent)

Theorem

Suppose M is a countable structure with a stationary independence relation compatible with a closure operation cl. Suppose that $G = \mathrm{Aut}(M)$ fixes every element of $\mathrm{cl}(\emptyset)$. If $g \in G$ moves almost maximally, then every element of G is a product of 16 conjugates of g.

REMARKS:

1. If $\mathrm{cl}(X) = X \ \forall X$, this is proved in the paper of Tent and Ziegler.
2. As observed by Tent and Ziegler, it implies the result of Macpherson and Tent for the free amalgamation example.
3. Proof is essentialy the same as the the Tent - Ziegler result.

Other uses of SIRs and variations:

- I. Kaplan and P. Simon (2017): Canonical independence relations
- Yibei Li (2018) - examples where we can drop the symmetry requirement in the SIR and still use the Tent - Ziegler method to prove simplicity of $\mathrm{Aut}(M)$.

4. Topological Dynamics

G a topological group.

G-flow: compact, Hausdorff, non-empty space X with a continuous G-action.

A G-flow X is *minimal* if every G-orbit on X is dense.

FACT: (Ellis) There is a unique *universal* minimal G-flow, $M(G)$.

[If X is any G-flow, there is a cts G-map $M(G) \to X$.]

QUESTION: Can we describe $M(G)$?

REMARKS: (1) If G is locally compact but not compact, the answer is 'not in a meaningful way';
(2) For many familiar homogeneous $\mathbb{M}$, $M(\mathrm{Aut}(\mathbb{M}))$ can be described.

NOTATION: In this part of the talk $\mathbb{M}$ denotes a structure with domain M etc

G-flows

$G = \mathrm{Aut}(\mathbb{M})$. Some G-flows:

1. Consider $Y = \{0,1\}^{M^n}$ as a G-flow.
   Also consider G-invariant, closed subspaces X of Y.
2. G-invariant, closed subspaces of $St(\mathbb{M})$, Stone space over M.

EXAMPLES: (1) $G = \mathrm{Sym}(M)$. We have a G-flow:

$$LO(M) = \{R \subseteq M^2 : R \text{ is a linear order on } M\}.$$

This is minimal.

(2) Let $\mathbb{P}$ be the Fraïssé limit of the class of all finite partial orders. As above, $LO(P)$ is an $\mathrm{Aut}(\mathbb{P})$-flow. But it is not minimal - the linear orderings which extend the ordering on $\mathbb{P}$ form a subflow.

Extreme amenability

Definition

Suppose G is a topological group.

1. G is *amenable* if every G-flow X supports a G-invariant Borel probability measure.
2. G is *extremely amenable* if every G-flow has a fixed point.

EXAMPLE: Suppose $H \leq \mathrm{Sym}(M)$ is e.a. Then H fixes a linear ordering on M.

THEOREM: (1) (Pestov, 1998) $\mathrm{Aut}((\mathbb{Q}; \leq))$ is e.a.
(2) (Glasner, Weiss, 2002) The universal minimal flow of $\mathrm{Sym}(M)$ is $LO(M)$.

Note that as a corollary to (2) we can see that $\mathrm{Sym}(M)$ is amenable.

REMARK: (Kechris and Rosendal) If $\mathbb{M}$ is the Fraïssé limit of a class of finite structures with EPPA, then $\mathrm{Aut}(\mathbb{M})$ is amenable.

The Kechris - Pestov - Todorčević Correspondence

Theorem (KPT, 2005)

Suppose $\mathbb{M}$ is a countable, homogeneous, linearly ordered relational structures with age $\mathcal{A}$. TFAE:

1. $\mathrm{Aut}(\mathbb{M})$ is extremely amenable.
2. $\mathcal{A}$ is a Ramsey class.

So Ramsey classes correspond to homogeneous structures with e.a. automorphism groups.

Ramsey classes

$L^{\leq}$: relational language with $\leq$.
$\mathcal{A}$: a class of finite $L^{\leq}$-structures closed under substrs and satisfying JEP and where $\leq$ is a linear ordering.

DEFINITION: Say that $\mathcal{A}$ is a Ramsey class if whenever $A \subseteq B \in \mathcal{A}$, there is $B \subseteq C \in \mathcal{A}$ such that if

$$\gamma : \binom{C}{A} \to \{0, 1\}$$

is a 2-colouring of the copies of A in C, there is $B' \in \binom{C}{B}$ (a copy of B in C) such that γ is constant on $\binom{B'}{A}$.

EXAMPLES: (1) $L = \{\leq\}$. Take $\mathcal{A} =$ finite linear orders.
(2) (Nešetřil - Rödl) The class $\mathcal{G}^{\leq}$ of linearly ordered finite graphs.

THEOREM: (Nešetřil) If $\mathcal{A}$ is a Ramsey class, then $\mathcal{A}$ has the amalgamation property. [Notation: Denote the Fraïssé limit by $\mathbb{F}(\mathcal{A})$.]

COR: The automorphism group of $\mathbb{F}(\mathcal{G}^{\leq})$ is e.a.

Computing $M(G)$ (KPT, 2005; Nguyen Van Thé 2013)

- L relational;
- $\mathbb{M}$ a countable homogeneous L-structure;
- $G = \mathrm{Aut}(\mathbb{M})$;
- $\mathcal{A}$ the age of $\mathbb{M}$.

DEF: Say closed $H \leq G$ is *coprecompact* (in G) if for every G-orbit $\Delta \subseteq M^n$, H has finitely many orbits on Δ.

ASSUME: G has a closed coprecompact extremely amenable subgroup H.

Think of H as $\mathrm{Aut}(\mathbb{M}^*)$ for some homogeneous L^*-structure $\mathbb{M}^*$, where $L^* \supseteq L$ is relational. Let $\mathcal{A}^*$ be the age of $\mathbb{M}^*$.

NOTE: Coprecompactness of H in G means that each $C \in \mathcal{A}$ has finitely many expansions in $\mathcal{A}^*$.

Computing $M(G)$ ....

DEF: Let $X(\mathcal{A}^*)$ be the set of expansions of $\mathbb{M}$ to L^*-structures in which the induced structure on each finite subset is in $\mathcal{A}^*$.
Topology: basic open set - all expansions agreeing on a particular finite subset.

– This is a G-flow. Moreover:

1. Any minimal subflow of $X(\mathcal{A}^*)$ is isomorphic to $M(G)$;
2. The G-orbit containing $\mathbb{M}^*$ is comeagre in $X(\mathcal{A}^*)$;
3. H may be chosen so that $X(\mathcal{A}^*)$ is minimal;
4. Every minimal G-flow has a comeagre orbit.
5. $M(G)$ is metrizable.

EXAMPLE: If $\mathbb{M}$ is the random graph, then $M(G) = LO(\mathbb{M})$.

The converse

Work of Zucker, Melleray - Nguyen Van Thé - Tsankov; Ben Yaacov - Melleray - Tsankov (2014-15) shows there is a converse:

If $G = \mathrm{Aut}(\mathbb{M})$ has a metrizable universal minimal flow, then G has a coprecompact closed extremely amenable subgroup.

Question

- Question asked (around 2011) by: Bodirsky, Pinsker, Tsankov; Nešetřil; Nguyen Van Thé:
  - If $\mathbb{M}$ is countable ω-categorical, is there an ω-categorical expansion $\mathbb{M}^*$ of $\mathbb{M}$ with $\mathrm{Aut}(\mathbb{M}^*)$ extremely amenable? Equivalently, is there a coprecompact e.a. closed subgroup of $\mathrm{Aut}(\mathbb{M})$?
- Next part: An example where this does not happen.
- Particularly interesting case: $\mathbb{M}$ homogeneous in a finite relational language - still open.

Sparse graphs.

DEF: Suppose $k \in \mathbb{N}$. A graph $M = (M; E)$ is *k-sparse* if for all finite $A \subseteq M$ we have $|E[A]| \leq k|A|$.

FACT: If the graph $M = (M; E)$ is k-sparse, then it is *k-orientable*: the edges of M can be directed so that each vertex has at most k directed edges coming out.

DEF: If M is k-sparse, let

$$X(M) = \{D \subseteq M^2 : (M; D) \text{ is a } k\text{-orientation of } M\} \subseteq \{0,1\}^{M^2}.$$

Note that this is an $\mathrm{Aut}(M)$-flow.

Theorem A

FACT: (Hrushovski) There is an ω-categorical 2-sparse graph M_F with all vertices of infinite valency.

Theorem A (DE, Jan Hubička and Jaroslav Nešetřil)

Suppose M is a countable, k-sparse graph of infinite valency. If $H \leq \mathrm{Aut}(M)$ is amenable, then H has infinitely many orbits on M^2.

COROLLARY: There is no coprecompact amenable subgroup of $\mathrm{Aut}(M_F)$.

Proof of Thm A: Step 1

- Suppose M is a graph with all vertices of infinite valency and $H \leq \mathrm{Aut}(M)$ has finitely many orbits on M^2.
- If $c \in M$ let H_c denote the stabilizer of c in H.
- For $c \in M$ let $\mathrm{cl}(c)$ be the union of the finite H_c-orbits on M.
- There is $n \in \mathbb{N}$ s.t. $|\mathrm{cl}(c)| \leq n$ for all $c \in M$.
- If $b \in \mathrm{cl}(c)$ then $\mathrm{cl}(b) \subseteq \mathrm{cl}(c)$.
- STEP 1: There are adjacent $a, b \in M$ such that b is in an infinite H_a-orbit and a is in an infinite H_b-orbit.

PROOF: Suppose there do not exist such a, b. Then for every edge a, b in M either $a \in \mathrm{cl}(b)$ or $b \in \mathrm{cl}(a)$. Take b with $\mathrm{cl}(b)$ of maximal size. There is $a \notin \mathrm{cl}(b)$ adjacent to b. By assumption, $\mathrm{cl}(a) \supset \mathrm{cl}(b)$: contradiction.

Proof of Thm A: step 2

- GIVEN: M is a k-sparse graph, $H \leq \mathrm{Aut}(M)$, and $a, b \in M$ are adjacent and such that a in an infinite H_b-orbit and b is in an infinite H_a-orbit.
- Show H is not amenable.
- Suppose there is an H-invariant probability measure μ on $X(M)$.
- Let $S(ab) = \{D \in X(M) : (a, b) \in D\}$. May assume $p = \mu(S(ab)) > 0$.
- Let $b_1, \ldots, b_n$ be in the same H_a-orbit as b and s_i the characteristic function of $S(ab_i)$. Note $\mu(S(ab_i)) = p$.
- For $D \in X(M)$,

$$\sum_{i \leq n} s_i(D) \leq k \text{ so } \int_{D \in X(M)} \sum_{i \leq n} s_i(D) d\mu(D) \leq k.$$

- So $np \leq k$: contradiction.

Further result

THEOREM B: Suppose $Y \subseteq X(\mathrm{Aut}(M_F))$ is a minimal $\mathrm{Aut}(M_F)$-subflow. Then all $\mathrm{Aut}(M_F)$-orbits on Y are meagre in Y.

QUESTION: (Tsankov) Is there a non-trivial minimal $\mathrm{Aut}(M_F)$-flow with a comeagre orbit?

Open Questions

QUESTION: (Bodirsky, . . .) If $\mathbb{M}$ is a structure homogeneous for a finite relational language, is there a coprecompact e.a. subgroup $H \leq \mathrm{Aut}(\mathbb{M})$?

SIDE QUESTION: Is there a homogeneous structure in a finite relational language in which a sparse graph of infinite valency can be interpreted?

QUESTION: (A. Ivanov) If $\mathbb{M}$ is ω-categorical and $\mathrm{Aut}(\mathbb{M})$ is amenable, is there a coprecompact e.a. subgroup $H \leq \mathrm{Aut}(\mathbb{M})$?

Appendix: Hrushovski's construction (1)

- $\mathcal{G}$: class of finite graphs $(A; R)$
- If $C \subseteq A \in \mathcal{G}$ let
$$\delta(C) = 2|C| - |R[C]|.$$
(Predimension of C.)
- If $A \subseteq B \in \mathcal{C}$ write $A \leq_d B$ if $\delta(X) > \delta(A)$ whenever $A \subset X \subseteq B$.
- Note: If $A \leq_d B \leq_d C$ then $A \leq_d C$.

Hrushovski's construction (2)

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ an increasing function which tends to infinity.
- Let
$$\mathcal{G}_F = \{A \in \mathcal{G} : \delta(Y) \geq F(|Y|) \text{ for all } Y \subseteq A\}.$$

- For suitable F the class $(\mathcal{G}_F, \leq_d)$ has free amalgamation over $\leq_d$-substructures.
- In this case the Fraïssé limit construction gives a countable graph M_F characterised by:
  - M_F is the union of a chain of finite $\leq_d$-subgraphs;
  - every graph in $\mathcal{G}_F$ is isomorphic to a $\leq_d$-subgraph of M_F;
  - isomorphisms between finite $\leq_d$-subgraphs of M_F extend to automorphisms.
- The graph M_F is 2-sparse and ω-categorical.