Invariant random subgroups III

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1097th Anniversary of the Treaty of Bonn

Pop Quiz

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Problem

Classify the ergodic IRSs of Hall's universal locally finite group.

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Theorem

Hall's universal locally finite group U is character rigid.

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- Let χ be a character of *U*.

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- There exists $r \in [\,0, 1\,]$ such that $\chi \restriction H = r\chi_{\text{con}} + (1 - r)\chi_{\text{reg}}$.

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- Thus $\chi(g) = \chi(h) = r$.
- Hence $\chi = r\chi_{\text{con}} + (1 - r)\chi_{\text{reg}}$.

Indecomposable characters of $L(\mathrm{Alt})$-groups

Definition

If ν is an IRS of the countable group G, then the associated character is defined by $\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \})$.

Theorem (Thomas 2018)

If G is an $L(\mathrm{Alt})$-group and $G \not\cong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are precisely the associated characters of its ergodic IRSs.

Indecomposable characters of $L(\mathrm{Alt})$-groups

The Indecomposability Problem

Prove that if $G \ncong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-groups and ν is an ergodic IRS of G, then the associated character χ_ν is indecomposable.

Remark

It is known that χ_{con}, χ_{reg} are indecomposable and so we can suppose that ν is a nontrivial ergodic IRS.

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The Realizability Problem

Prove that if $G \ncong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-groups and χ is an indecomposable character, then there exists an ergodic IRS ν of G such that $\chi = \chi_\nu$.

The Indecomposability Problem

Definition

If $G \curvearrowright (Z, \mu)$ *is an ergodic action, then the associated character is*

$$\chi(g) = \mu(\mathrm{Fix}_Z(g)).$$

Problem

Find necessary and sufficient conditions for the associated character of an ergodic action $G \curvearrowright (Z, \mu)$ to be indecomposable.

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Definition (Vershik)

An ergodic action $G \curvearrowright (Z, \mu)$ is said to be extremely non-free if the map $z \mapsto G_z$ is μ-a.e. injective.

A Sufficient Condition for Indecomposability

Notation

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Definition (Dudko-Grigorchuk)

A measure-preserving action $G \curvearrowright (Z, \mu)$ is said to be perfectly non-free if there exists a collection $\mathcal{A} \subseteq \mathcal{M}$ such that:

(i) *$\mathcal{A} \cup \mathcal{N}$ generates sigma-algebra $\mathcal{M}$;*

(ii) *for each $A \in \mathcal{A}$, the orbits $G_{(X \smallsetminus A)} \cdot z$ are infinite for μ-a.e. $z \in A$.*

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Theorem (Dudko-Grigorchuk 2018)

If $G \curvearrowright (Z, \mu)$ is an ergodic, perfectly non-free action of a countable group on a standard probability space, then the character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is indecomposable.

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If $G \ncong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-groups and ν is a nontrivial ergodic IRS of G, then ν is the stabilizer distribution of an ergodic, perfectly non-free action $G \curvearrowright (Z, \mu)$.

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Corollary (Thomas 2018)

If $G \ncong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-groups and ν is an ergodic IRS of G, then the associated character χ_ν is indecomposable.

A Decomposable Character of $\mathrm{Alt}(\mathbb{N})$

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If χ is a character of $\mathrm{Alt}(\mathbb{N})$, then the following are equivalent:

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- For each $\xi \in 2^{\mathbb{N}}$ and $i = 0, 1$, let $B_i^{\xi} = \{\, n \in \mathbb{N} \mid \xi(n) = i \,\}$.
- Let $\varphi : 2^{\mathbb{N}} \to \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N}}$ be the $\mathrm{Alt}(\mathbb{N})$-equivariant map defined by $\varphi(\xi) = \mathrm{Alt}(B_0^{\xi}) \times \mathrm{Alt}(B_1^{\xi})$.

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- Then $\nu = \varphi_* \mu$ is an ergodic IRS of $\mathrm{Alt}(\mathbb{N})$.

The associated character is

$$\chi_\nu(g) = \mu(\{\xi \in 2^{\mathbb{N}} \mid g \in \mathrm{Alt}(B_0^\xi) \times \mathrm{Alt}(B_1^\xi)\});$$

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$$\chi_\nu((\,1\,2\,)(\,3\,4\,)) = 1/2^4 + 1/2^4 = \chi_\nu((\,5\,6\,)(\,7\,8\,));$$

while, on the other hand,

$$\chi_\nu((\,1\,2\,)(\,3\,4\,)(\,5\,6\,)(\,7\,8\,)) = \frac{\binom{4}{0} + \binom{4}{2} + \binom{4}{4}}{2^8} = 1/2^5.$$

The Main Theorem Revisited

Definition

- G is an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite alternating groups G_i.
- Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.

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If G is an $L(\mathrm{Alt})$-group and $G \ncong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are precisely the associated characters of its ergodic IRSs.

Thoma's Classification

If $G = \mathrm{Alt}(\mathbb{N})$, $\mathrm{Fin}(\mathbb{N})$, then the indecomposable characters of G are precisely the functions $\chi_{(\boldsymbol{\alpha};\boldsymbol{\beta})} : G \to \mathbb{C}$ such that there exist two sequences $\boldsymbol{\alpha} = (\, \alpha_i \mid i \in \mathbb{N}^+ \,)$ and $\boldsymbol{\beta} = (\, \beta_i \mid i \in \mathbb{N}^+ \,)$ of non-negative real numbers satisfying

- $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \geq 0\,;$
- $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_i \geq \cdots \geq 0\,;$
- $\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq 1;$

such that

$$\chi_{(\boldsymbol{\alpha};\boldsymbol{\beta})}(g) = \prod_{n=2}^{\infty} \Big(\sum_{i=1}^{\infty} \alpha_i^n + (-1)^{n+1} \sum_{i=1}^{\infty} \beta_i^n \Big)^{c_n(g)},$$

where $c_n(g)$ is the number of cycles of length n in the cyclic decomposition of the permutation g.

A non-realizable indecomposable character

Fact

Thoma's classification of the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$ *includes*

$$\chi(g) = \prod_{n=2}^{\infty} \left((1/2)^n + (-1)^{n+1}(1/2)^n \right)^{c_n(g)},$$

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Example

Thus $\chi(\,(\,a\,b\,c\,)\,) = 1/4$ and $\chi(\,(\,a\,b\,)(\,c\,d\,)\,) = 0$.

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- Suppose that ν is an ergodic IRS such that

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- But $(1\,2\,n)(1\,2\,m) = (1\,n)(2\,m)$ and

$$\nu(\{ H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \mid (1\,n)(2\,m) \in H \}) = 0.$$

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One of Vershik's many insights

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- Then $\mathrm{Fin}(\mathbb{N}) \curvearrowright (\mathbb{N}^{\mathbb{N}}, \mu_\alpha)$ acts ergodically via the shift action $(\pi \cdot \xi)(n) = \xi(\pi^{-1}(n))$.

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The ergodic IRSs of $\mathsf{Fin}(\mathbb{N})$

- Let $I = \{ i \in \mathbb{N}^+ \mid \alpha_i > 0 \}$ and let $S_\alpha = \bigoplus_{i \in I} C_i$, where each $C_i = \{ \pm 1 \}$ is cyclic of order 2.

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- Let ξ be μ_α-random and let s_ξ be the homomorphism

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- Let $\xi \overset{f^A}{\mapsto} H_\xi = s_\xi^{-1}(A)$.
- Then $\nu_\alpha^A = (f^A)_* \mu_\alpha$ is an ergodic IRS of $\mathrm{Fin}(\mathbb{N})$.

Example: the "Maximal" IRSs

Observation

Note that if $A = S_\alpha$, then

$$\chi_{\nu_\alpha^{S_\alpha}}(g) = \mu_\alpha(\{\, \xi \in \mathbb{N}^\mathbb{N} \mid g \in \bigoplus_{i \in I} \mathrm{Fin}(B_i^\xi) \,\})$$

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Corollary

$\chi_{\nu_\alpha^{S_\alpha}}$ *is indecomposable.*

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Theorem (Thomas-Tucker-Drob 2016)

If $A \neq S_\alpha$, then $\chi_{\nu_\alpha^A}$ is not indecomposable.

The Dual Group $\widehat{S}_\alpha$ enters the picture ...

- Let $\widehat{S}_\alpha$ be the compact group of homomorphisms

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Definition

If $\sigma \in \widehat{S}_\alpha$, then χ_α^σ is the indecomposable character defined by

$$\chi_\alpha^\sigma(g) = \prod_{n=2}^{\infty} \Big(\sum_{i \in I} \sigma(i)^{n+1} \alpha_i^n \Big)^{c_n(g)}.$$

The decomposition of the associated character

For each subgroup $A \leqslant S_\alpha$, let

$$(\widehat{S_\alpha / A}) = \{ \sigma \in \widehat{S}_\alpha \mid \sigma(a) = 1 \text{ for all } a \in A \}$$

and let μ_α^A be the Haar probability measure on $(\widehat{S_\alpha / A})$.

Theorem (Thomas, October 2018)

If α, A are as above, then

$$\chi_{\nu_\alpha^A} = \int_{\sigma \in (\widehat{S_\alpha / A})} \chi_\alpha^\sigma \, d\mu_\alpha^A(\sigma).$$

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The Natural Reaction

Very pretty ... but what does it mean?

Asymptotic decompositions of permutation characters

Suppose that $n = A_1 \sqcup \cdots \sqcup A_t$ and that

$$\bigoplus_{1 \leq \ell \leq t} \mathrm{Alt}(A_\ell) \leqslant H \trianglelefteq M = \bigoplus_{1 \leq \ell \leq t} \mathrm{Sym}(A_\ell) \leq S = \mathrm{Sym}(n).$$

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With the above assumptions, we have that

$$\begin{aligned} \frac{1_H^S}{1_H^S(1)} &= \frac{1}{[S:H]} \left(1_H^M \right)^S \\ &= \sum_{\theta \in \mathrm{Irr}(M/H)} \frac{1}{[M:H]} \frac{\theta^S}{\theta^S(1)}. \end{aligned}$$

An easy example ...

- Suppose that $n = A_1 \sqcup A_2$ and that

$$H = \mathrm{Sym}(A_1) \oplus \mathrm{Alt}(A_2) \trianglelefteq M = \mathrm{Sym}(A_1) \oplus \mathrm{Sym}(A_2) \leqslant S.$$

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- Thus we have that

$$\frac{1_H^S}{1_H^S(1)} = \frac{1}{2}\ \frac{1_M^S}{1_M^S(1)} + \frac{1}{2}\ \frac{\theta_M^S}{\theta_M^S(1)}.$$

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- Note that if $g \in S$ is a k-cycle, then

$$\frac{\theta_M^S(g)}{[\,S : M\,]} = \frac{|\{\, s \in S \mid sgs^{-1} \in \mathrm{Sym}(A_1)\,\}|}{|S|} + \mathrm{sgn}(g)\frac{|\{\, s \in S \mid sgs^{-1} \in \mathrm{Sym}(A_2)\,\}|}{|S|}$$

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- And thus we begin to understand Thoma's theorem ...

The Main Theorem Revisited

Definition

- G is an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite alternating groups G_i.
- Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.

Theorem (Thomas 2018)

If G is an $L(\mathrm{Alt})$-group and $G \ncong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are precisely the associated characters of its ergodic IRSs.

The Asymptotic Approach to Characters

Theorem (Vershik-Kerov 1985)

If $G = \bigcup_{i \in \mathbb{N}} G_i$ is the increasing union of the finite subgroups G_i and χ is an indecomposable character of G, then there exist characters θ_i of irreducible representations π_i of G_i such that

$$\chi(g) = \lim_{i \to \infty} \frac{\theta_i(g)}{\theta_i(1)}.$$

Remark

Here $\theta_i(g) = \mathrm{trace}(\, \pi_i(g)\,)$ is the "usual" character associated with an irreducible representation π_i of the finite group G_i.

Irreducible Characters of Finite Symmetric Groups

Notation

Let $\lambda = (\ell_1, \ell_2, \cdots, \ell_r)$ be a partition of n; i.e. a sequence of integers such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r > 0$ and $\ell_1 + \ell_2 + \cdots + \ell_r = n$.

- φ_λ denotes the corresponding irreducible character of $\mathrm{Sym}(n)$.

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- λ^* denotes the corresponding dual partition.
- $\leq$ denotes the lexicographic ordering on the partitions of n.

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Let $\theta_\lambda = \varphi_\lambda \restriction \mathrm{Alt}(n)$.

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- θ_λ *is irreducible iff* $\lambda^* \neq \lambda$.

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- *If $\lambda^* = \lambda$, then θ_λ is the sum of two distinct irreducible characters.*

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Conclusion

The irreducible characters are "almost" parameterized by the partitions λ such that $\lambda \geq \lambda^*$.

An Upper Bound on the Irreducible Characters

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Theorem (Roichman)

There exist constants $b > 0$ and $0 < q < 1$ such that for sufficiently large n, for every partition $\lambda \geq \lambda^*$ of n and every $g \in \mathrm{Alt}(n)$,

$$\left| \frac{\theta_\lambda(g)}{\theta_\lambda(1)} \right| \leq \left(\max \left\{ q, \frac{n - d(\lambda)}{n} \right\} \right)^{b \cdot |\operatorname{supp}(g)|}.$$

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Full Limits

Definition

Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$.

- The embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is full if $\mathrm{Alt}(\Delta_i)$ has no trivial orbits on Δ_{i+1}.
- G is a full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ if every embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is full.

Lemma (Thomas-Tucker-Drob 2016)

If $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, then $\liminf |\mathrm{supp}_{\Delta_i}(g)|/|\Delta_i| > 0$ for all $1 \neq g \in G$.

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Remark

If $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-group, then we can express $G = \bigcup_{\ell \in \mathbb{N}} G(\ell)$ as an increasing union of full limits $G(\ell)$.

Sketch Proof for Full Limits

- Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and that $\chi \neq \chi_{\mathrm{reg}}, \chi_{\mathrm{con}}$ is a nontrivial indecomposable character of G.

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- By Vershik-Kerov, there exist partitions λ_i of $n_i = |\Delta_i|$ such that

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- Since $\chi \neq \chi_{\mathrm{reg}}$, $d(\lambda_i)$ is bounded as $i \to \infty$.
- Hence there exists an infinite $I \subseteq \mathbb{N}$ and a fixed $(\ell_2, \cdots, \ell_r)$ such that $\lambda_i = (n_i - d(\lambda_i), \ell_2, \cdots, \ell_r)$ for all $i \in I$.

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- Let $d = d(\lambda_i)$ and let π_{λ_i} be the permutation character corresponding to the action of $G_i = \mathrm{Alt}(\Delta_i)$ on the partitions $\Delta_i = P_1 \sqcup \cdots \sqcup P_r$ with $|P_k| = \ell_k$ for $2 \leq k \leq r$.

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- Hence if $\Omega_i = \Delta_i^d$ and $G_i \curvearrowright \Omega_i$ is the diagonal action, then

$$\chi(g) = \lim_{i \in I} \frac{|\operatorname{Fix}_{\Omega_i}(g)|}{|\Omega_i|}.$$

The Loeb Measure Construction

- For each $i \in I$, let μ_i be the uniform probability measure on Ω_i.
- Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $I \in \mathcal{U}$ and let $(\, X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}} \,)$ be the corresponding Loeb probablity space.

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- Let $f : X_{\mathcal{U}} \to \mathrm{Sub}_G$ be the G-equivariant $\mathbf{B}_{\mathcal{U}}$-measurable map defined by $x \mapsto G_x$.
- Then $\nu = f_* \, \mu_{\mathcal{U}}$ is an IRS of G; and for each $g \in G$,

$$\chi(g) = \lim_{\mathcal{U}} \frac{|\,\mathrm{Fix}_{\Omega_i}(g)|}{|\Omega_i|} = \mu_{\mathcal{U}}(\mathrm{Fix}_{X_{\mathcal{U}}}(g))$$

The Loeb Measure Construction

- For each $i \in I$, let μ_i be the uniform probability measure on Ω_i.
- Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $I \in \mathcal{U}$ and let $(X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the corresponding Loeb probablity space.
- Then $G \curvearrowright (X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$.
- Let $f : X_{\mathcal{U}} \to \mathrm{Sub}_G$ be the G-equivariant $\mathbf{B}_{\mathcal{U}}$-measurable map defined by $x \mapsto G_x$.
- Then $\nu = f_* \mu_{\mathcal{U}}$ is an IRS of G; and for each $g \in G$,

$$\chi(g) = \lim_{\mathcal{U}} \frac{|\mathrm{Fix}_{\Omega_i}(g)|}{|\Omega_i|} = \mu_{\mathcal{U}}(\mathrm{Fix}_{X_{\mathcal{U}}}(g)) = \nu(\{ H \in Sub_G \mid g \in H \}).$$

Open Problems: Strong Simplicity

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Theorem (Peterson-Thom 2013 & Thomas-Tucker-Drob 2016)

If the countably infinite simple group G is character rigid, then G is strongly simple.

Question

Does there exist a strongly simple (locally finite) group which is not character rigid?

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Theorem (Peterson-Thom 2013 & Thomas-Tucker-Drob 2016)

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Does there exist a strongly simple (locally finite) group which is not character rigid?

Observation

The classes of strongly simple groups and character rigid simple groups are both co-analytic.

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Does there exist a strongly simple (locally finite) group which is not character rigid?

Theorem

The class of character rigid simple locally finite groups is Borel.

Open Problems: Strong Simplicity

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If the countably infinite simple group G is character rigid, then G is strongly simple.

Question

Does there exist a strongly simple (locally finite) group which is not character rigid?

Question

Is the class of strongly simple locally finite groups Borel?

Definition

If $G \curvearrowright (Z, \mu)$ is an ergodic action, then the associated character is

$$\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,).$$

Problem

Find necessary and sufficient conditions for the associated character of an ergodic action $G \curvearrowright (Z, \mu)$ to be indecomposable.

Open Problems: The Classifiability Problems

Problem

If G is an arbitrary countable simple locally finite group, do there necessarily exist "nice parametrizations" of the ergodic IRSs of G and of the indecomposable characters of G?

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The End