"Answer from Donin–Kulish–Mud

Def^n (Majid) The quantum alg. $\mathcal{O}_q(G)$ is

$\mathcal{O}_q(G) \xrightarrow{REA} \frac{2}{q}\big|_{q=1}$, $q \to 1$, $\mathcal{O}(G)$, $U(\mathfrak{g})$, $S(\mathfrak{g}^*) \cong S(\mathfrak{g})$

$$\mathcal{O}(G) \cong \bigoplus_{X \text{ simple}} X^* \otimes X$$

$(X \otimes Y)^*$

$Y^* \otimes X^*$, $e X \otimes Y$

$X^* \otimes X \otimes Y^* \otimes Y$

def^n's an alg. str. on $\mathcal{O}_q(G)$

$V^* \otimes W \longrightarrow \mathcal{O}(G)$

$f \otimes v \mapsto \varphi_{f,v}(g) = f(gv)$

$a^2_1 a^1_2 = q^2 a^1_2 a^2_1 + (1-q^{-2})(a^1_1 - a^2_2)a^2_2$

$a^i_j = \delta^i_j + t_h E^i_j$

$q^2 = e^{t_h}$ $\quad E^1_1 E^2_1 = E^2_1 E^1_1 + \underbrace{E^1_1 - E^2_2}_{[E^1_1, E^2_1]}$

$GL_2(\mathbb{R}^2)$ $\mathcal{O}_q(GL_2) = \mathbb{C}\langle a^1_1, a^1_2, a^2_1, a^2_2 \rangle (\det_q^{-1})$ / rels

Notice $B_n(S)$ receives maps $B_n(\mathbb{R}^2) \longrightarrow B_n(S)$

Notice $B_n(S)$ receives maps $B_n(\mathbb{R}^2) \longrightarrow B_n(S)$

Can we define a category $Z_A(S)$ associated to S extending the $B_n(\mathbb{R}^2)$ to S in a universal way?

Thm (D-K-M). Consider $M \in \mathcal{O}_q(G)$-module $V \in \mathrm{Rep}_q G$.

$$\underbrace{M \otimes V} \otimes \cdots \otimes V \circlearrowleft B_n(\mathrm{Ann})$$

$$\mathbb{X} \in \mathcal{O}_q(G) \otimes U_q(\mathfrak{g})$$

$$X_1 = \mathbb{X} \otimes \mathrm{id}_{V^{n-1}}$$

$B_n(\mathrm{Ann})$-modules are obtained by pullback.

$\mathbb{X}$ compatible w/ $V, W \rightsquigarrow V \otimes W$.

Any other alg A w/ all this structure $\mathcal{O}_q(G) \xrightarrow{\exists} A$

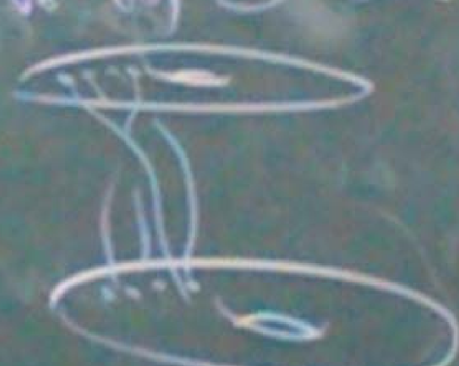
III) Surface braid groups

IV) An answer and examples.

Q: $\exists$ categories $Z_A(S)$ w/

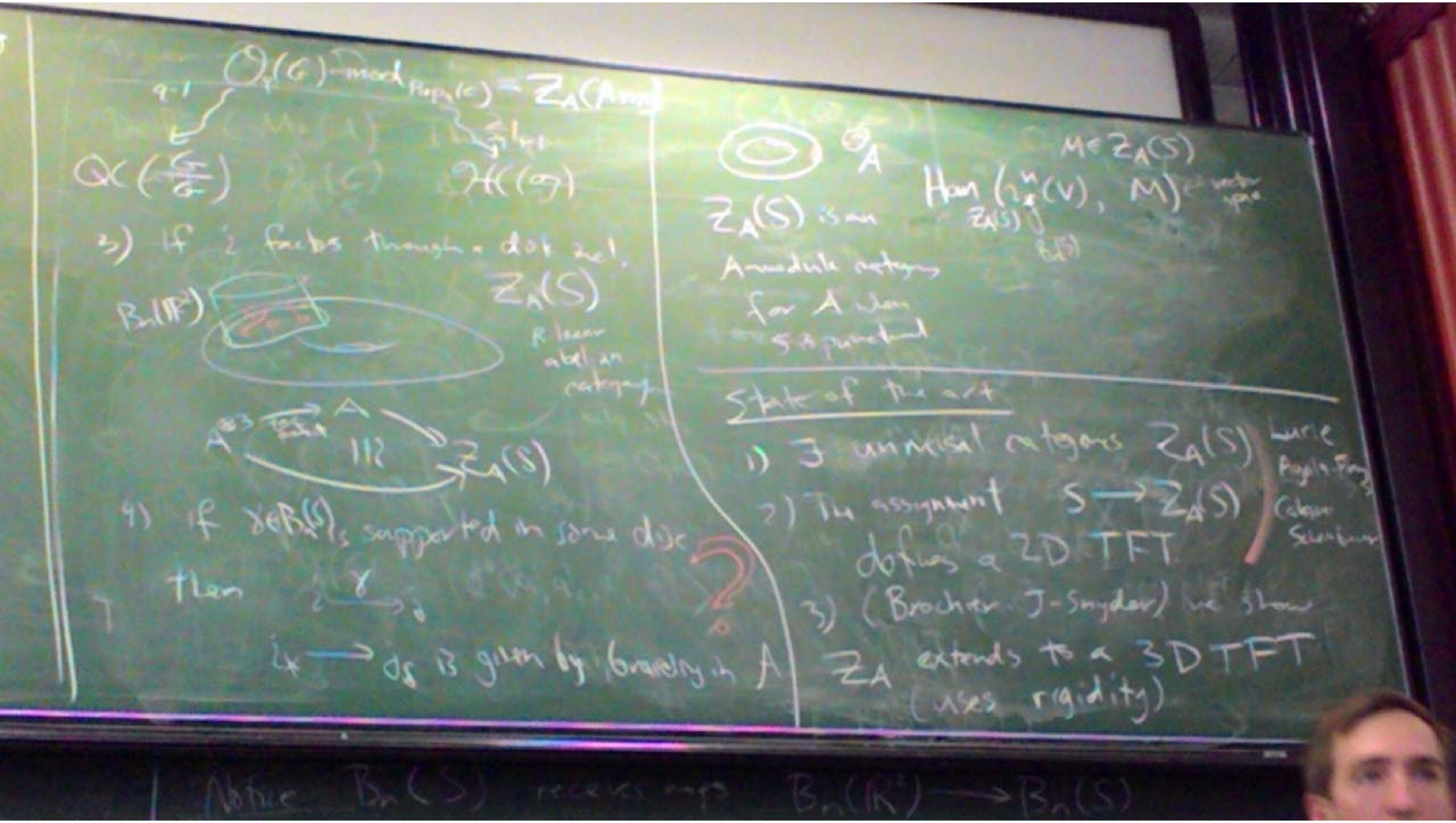
1) $i: \underbrace{D \sqcup \cdots \sqcup D}_{n} \longrightarrow S$ $\quad i_A^{\boxtimes n}(-): A^{\boxtimes n} \longrightarrow Z_A(S)$

$i_A: A^{\boxtimes n} \longrightarrow Z_A(S)$ $\quad i_A(V_1 \boxtimes \cdots \boxtimes V_n) \circlearrowleft B_n(S)$



2) Gen an isotopy $i \xRightarrow{\gamma} j$

$A^{\boxtimes n} \underset{j_A}{\overset{i_A}{\rightrightarrows}} Z_A(S)$


$\mathcal{O}_q(G)$-mod$_{\mathrm{Rep}_q(G)} = Z_A(\mathrm{Ann})$
q→1
$QC\left(\frac{G}{G}\right)$
$\mathcal{H}((\mathfrak{g}))$
3) If i factors through a disk rel.
$B_n(\mathbb{R}^2)$
$Z_A(S)$
R-linear abelian category
$A^{\otimes 3}$
A
$Z_A(S)$
4) If $\gamma \in B_n(S)$ is supported in some disc
then
$i_* \longrightarrow \sigma_\gamma$ is given by braiding in A
$\mathcal{O}_A$
$M \in Z_A(S)$
$Z_A(S)$ is an
A-module category
for A when
S is punctured
Hom (V, M) ← vector space
State of the art
1) ∃ universal categories $Z_A(S)$
Lurie
2) The assignment $S \longrightarrow Z_A(S)$
defines a 2D TFT
3) (Brochier - J - Snyder) we show
Z_A extends to a 3D TFT
(uses rigidity)
Notice $B_n(S)$ receives maps $B_n(\mathbb{R}^2) \longrightarrow B_n(S)$

4) S is punctured

$$Z_A(S) = a_S\text{-mod}_{\mathrm{Rep}_q G}$$

5) a_S can be presented

6) If S is closed $Z_A(S) =$ "Ham red of $Z_A(S^\circ)$"

7) $Z_A(S)$ are def. quantizations of Atiyah-Bott symp. str. on $Ch_G(S)$

I) ~~Factorization homology + homotopy theory~~

II) Quantum groups + br. groups

III) Surface braid groups + a question

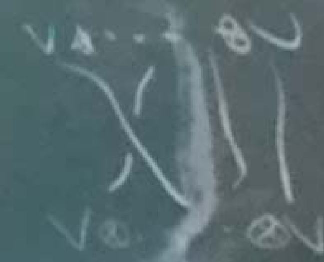
IV) An answer and examples.

$\mathcal{U} = \mathcal{U}_q(\mathfrak{g})$ quantum gp ass to some s.s. alg $G/\mathbb{C}$

$\rightsquigarrow \mathcal{U}(\mathfrak{g})$ = universal env. algebra.

$A = \mathrm{Rep}_q G$ of integrable $\mathcal{U}_q(\mathfrak{g})$ reps.

$V, W \in A \rightsquigarrow \sigma_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$



$B_n(\mathbb{R}^2) \circlearrowright V^{\otimes n}$

$A =$

$\mathcal{A}$ = rigid braided tensor ca... $\otimes, \varepsilon)$

$V \rightsquigarrow V^*$

S = surface $\quad \mathrm{Conf}_n(S) = \{ X \subseteq S \mid |X| = n \}$

$B_n(S) = \pi_1(\mathrm{Conf}_n(S))$

$\mathbb{R}^2$

S^2

$B_n(\mathrm{Ann}) = \langle X_1, B_n(\mathbb{R}^2) \rangle / \mathrm{rel}^{ns} \cong B_n(\mathbb{R}^2) \ltimes (\oplus \mathbb{Z} X_i)$

$X_{i+1} = T_i X_i T_i$

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$B_n(\mathbb{R}^2) \twoheadrightarrow B_n(S^2)$

$T_1, \dots, T_{n-1}$

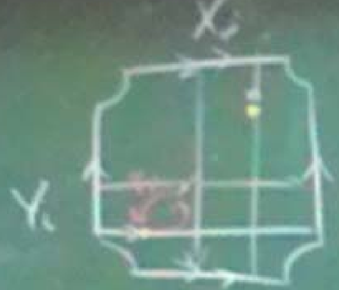
$T_1 T_2 \cdots T_{n-1}^2 \cdots T_2 T_1 = \mathrm{id}$

Refl^c equation

$T_1 X_1 T_1 X_1 = X_1 T_1 X_1 T_1$

Birman, Scott 2005

$B_n(T^2\setminus D^2)$



$$B_n(T^2\setminus D^2) = \underbrace{(\oplus \mathbb{Z} X_i) * B_n(\mathbb{R}^2)}_{\text{ABG}} * \underbrace{(\oplus \mathbb{Z} Y_i)}_{\text{ABG}} \Big/ \text{rel}^{ns}$$

$$* \quad Y_2 X_1 = X_1 Y_2 T_1^{-2} \qquad Y_2 = T_1 Y_1 T_1$$

$$X_1^{-1} Y_2^{-1} X_1 Y_2 = T_1^{-2}$$

$$B_n(T^2) = B_n(T^2\setminus D^2) \Big/ \langle \underbrace{Y_1 \cdots Y_n}_{\text{central}} \cdot X_i = X_i Y_1 \cdots Y_n \rangle$$

$$\mathbb{C}[B_n(T^2)] \longrightarrow \mathbb{H}_{q,t}(S_n)$$

$$(T_i - q^{-1}t)(T_i + q^{-1}t^{-1})$$