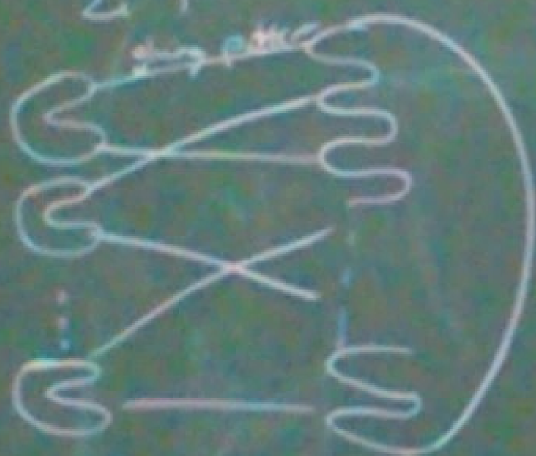
Representation Theory of Tensor Cats

LAST TIME: $\mathcal{A}$ rigid braided $\otimes$-cat, Σ surface $\xrightarrow[\text{homology}]{\text{factorization}}$ $\int_\Sigma \mathcal{A}$ a category that universally extends the B_n on $*\mathcal{A}$ to the $B_n(\Sigma)$-action on a $*$ on objects in

Σ - punctured surface

gluing pattern



handlebodies

eg

$\simeq$

$\simeq$

$$\mathcal{A}^{\boxtimes h} \circlearrowright \mathcal{A}^{\boxtimes 2h} \circlearrowleft \mathcal{A} \rightsquigarrow \mathcal{A}^{\boxtimes h} \boxtimes_{\mathcal{A}^{\boxtimes 2h}} \mathcal{A} \simeq \int_\Sigma \mathcal{A}$$

Thm (Ayala-Francis)

Frobenius Algebras

A f.d. $/k$

Frobenius ${}_A A \cong {}_A A^*$ $\longrightarrow$ $\beta : A \otimes A \longrightarrow A^* \otimes A \xrightarrow{\text{ev}} k$

Frobenius pairing

symmetric Frobenius ${}_A A_A \cong {}_A A^*_A$

$\boxed{\text{cool fact}}$ A symmetric Frob $\Rightarrow$ $HH^0(A) \xrightarrow{\cong} HH_0(A)$

center, cocenter

$\left(\begin{array}{l} HH_0 A = A \otimes_{A^e} A \\ HH_0 A^* = \text{traces on } A \end{array} \right)$

Ex G finite group, $k[G]$ is symm Frob

$HH^0(k[G]) = \text{span}\{\text{conj classes}\}$

non-canonical

$HH_0(k[G])^* = \text{span}\{\text{conj classes}\}$

M irred. A-module

1. $A \xrightarrow{act} \mathrm{End}(M)$

2.
$$\begin{array}{ccc} \mathrm{End}(M) & \xrightarrow{act^R} & A \\ \mathrm{Tr}\ \wr\Vert & & \wr\Vert \\ \mathrm{End}(M)^* & \xrightarrow{act^*} & A^* \end{array}$$

3. $e_M = act^R(act(1)) = act^R(id_M) \in A$

$$e_M^2 = \lambda e_M \qquad \Big(\beta(e_M, a) = \mathrm{Tr}_M(act(a)) \Big)$$
$$\lambda = \mathrm{Tr}_M(act(e_M))$$

4. $act^R\ \mathrm{End}\,M \longrightarrow A$ is a map of left A-modules

5. $\mathrm{End}(M) \xrightarrow{?} A e_M$

Rigid Tensor Categories

$(\ )^* : {}_{\mathcal{A}}\mathcal{A} \to {}_{\mathcal{A}}\mathcal{A}^{op}$ rigid

$(\ )^* : {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}_{\mathcal{A}}\mathcal{A}^{op}_{\mathcal{A}}$ pivotal

$\left(\begin{array}{c} \text{choice of} \\ (\ )^{**} \cong id \end{array} \right)$,

For any cat $\mathcal{A}$

$Hom : \mathcal{A}^{op} \times \mathcal{A} \to Vect$

$X, Y \mapsto Hom_{\mathcal{A}}(X, Y)$

Recall

$F : \mathcal{C} \rightleftarrows \mathcal{D} : F^R$ s.t. $Hom(FX, Y) \cong Hom(X, F^R Y)$.

Let $\mathcal{A}$ be a rigid tensor cat, $\mathcal{M}$ a right module cat

$\left(\begin{array}{l} \mathcal{M} \boxtimes \mathcal{A} \xrightarrow{\otimes} \mathcal{M} \\ m \boxtimes x \longrightarrow m \otimes x \end{array}, \begin{array}{l} m \otimes 1 \cong m \\ (m \otimes x) \otimes y \cong m \otimes (x \otimes y) \end{array} \right)$.

1. Choose a <u>progenerator</u> $m \in \mathcal{M}$, $act_m : \mathcal{A} \to \mathcal{M}$

$X \longmapsto m \otimes X$

s.t. (i) $act_m : \mathcal{A} \to \mathcal{M}$ is <u>dominant</u>

$\left(\forall n \in \mathcal{M} \ \exists x \in \mathcal{A} \text{ st. } 0 \to n \to act_m(x) \right)$

(ii) act_m^R preserves finite colimits.

2. $act_m^R : \mathcal{M} \to \mathcal{A}$ is <u>conservative</u>

$\left(act_m^R(x) \cong act_m^R(y) \Rightarrow x \cong y \right)$

3. $\mathrm{act}_m^R \circ \mathrm{act}_m(\mathbb{1}) \cong \mathrm{act}_m^R(m) =: \underline{\mathrm{End}}(m) \in \mathcal{A}$

$\underline{\mathrm{End}}(m) \otimes \underline{\mathrm{End}}(m) \longrightarrow \underline{\mathrm{End}}(m)$ map in $\mathcal{A}$

4. Rigid $\Rightarrow$ act_m^R is a functor of $\mathcal{A}$-module categories

Aside: $\underline{\mathrm{End}}(m)$ is an algebra in $\mathcal{A}$, left modules over $\underline{\mathrm{End}}(m)$ in $\mathcal{A}$ $\Big\}$ cat $\underline{\mathrm{End}}(m)\text{-mod}_{\mathcal{A}}$

$X \in \mathcal{A}$, $\underline{\mathrm{End}}(m) \otimes X \longrightarrow X$.

Thm (Beck's monadicity)

If m is a progenerator then $\mathcal{M} \cong \underline{\mathrm{End}}(m)\text{-mod}_{\mathcal{A}}$. $\boxtimes$

$n \in \mathcal{M}$, $\mathrm{act}_m^R(n) \in \mathcal{A}$ $\quad$ $\mathrm{act}_m^R(n) = \underline{\mathrm{Hom}}(m, n)$

act_m^R $\qquad$ $\circlearrowleft$ $\underline{\mathrm{End}}(m)$

Thm (Etz-B-J)

If $\mathcal{M}$ is an $\mathcal{A}$-module cat w/ progen m,

$T : \mathcal{A} \to \mathcal{B}$ dominant tensor functor

$$\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{B} \simeq T(\underline{\mathrm{End}}(m))\text{-mod}_{\mathcal{B}}$$

$\blacksquare$

Examples

(stupid example)

$\mathcal{A}$ as a right $\mathcal{A}$ module cat.

<u>claim</u> $\mathbb{1} \in \mathcal{A}$ is a progenerator

$\mathrm{id}_{\mathcal{A}} \cong \mathrm{act}_{\mathbb{1}} : \mathcal{A} \to \mathcal{A}$

$\mathrm{id}_{\mathcal{A}}^{R} = \mathrm{id}_{\mathcal{A}}$

$$\left.\right\} \; {}_{\mathcal{A}}\mathcal{A} \simeq \underline{\mathrm{End}}(\mathbb{1})\text{-mod}_{\mathcal{A}} \simeq \mathbb{1}\text{-mod}_{\mathcal{A}}$$

Recall: $\int_{\bigcirc} \mathcal{A} = \mathcal{A} \boxtimes_{\mathcal{A}^{\boxtimes 2}} \mathcal{A}$

$\mathcal{A} = \mathrm{Rep}_q G$

$\mathcal{A} \circlearrowleft \mathcal{A}^{\boxtimes 2}$

$M \cdot X \boxtimes Y = M \otimes X \otimes Y$

$(M \cdot X \boxtimes Y) \cdot V \boxtimes W = M \otimes X \otimes Y \otimes V \otimes W$

$M \cdot (X \boxtimes Y \cdot V \boxtimes W)$

$= M \cdot X \otimes V \boxtimes Y \otimes W$

$= M \otimes X \otimes V \otimes Y \otimes W$

braiding

claim: $\mathbb{1} \in \mathcal{A}$ is a progenerator

$act_{\mathbb{1}}(X \boxtimes \mathbb{1}) \cong \mathbb{1} \otimes X \otimes \mathbb{1} \cong X$

observe $act_{\mathbb{1}} \cong T : \mathcal{A}^{\boxtimes 2} \to \mathcal{A}$

what is

$\underline{\mathrm{End}(\mathbb{1})?}$

$\overset{\shortparallel}{T^R(\mathbb{1})}$

$$\mathrm{Hom}_{A^{\boxtimes 2}}\left(X \boxtimes Y,\ T^R(\mathbb{1})\right) \cong \mathrm{Hom}_A(X \otimes Y, \mathbb{1})$$

$$\cong \mathrm{Hom}_A(X, {}^*Y) \cong \mathrm{Hom}_A(Y, X^*)$$

$$\varphi \qquad\qquad \varphi^*$$

$\underline{1^{st}\text{ guess:}}$ $\bigoplus_X {}^*X \boxtimes X \in A^{\boxtimes 2}$

$$\begin{array}{l} X \boxtimes Y \xrightarrow{id \boxtimes \varphi^*} {}^*X \boxtimes X^* = {}^*(X^*) \boxtimes X^* \\ X \boxtimes Y \xrightarrow{\varphi \boxtimes id} {}^*Y \overset{\oplus}{\boxtimes} Y \end{array}$$

$\underline{\text{claim}}$ $T^R(\mathbb{1}) \cong \bigoplus {}^*X \boxtimes X \big/ \mathrm{Im}\left(\varphi \boxtimes id_X - id_Y \boxtimes \varphi^*\right)$

If A is semisimple $T^R(\mathbb{1}) \cong \bigoplus_{\text{simples}} {}^*X \boxtimes X$

$${}^*X \boxtimes X \cdot {}^*Y \boxtimes Y = {}^*X \otimes {}^*Y \boxtimes X \otimes Y \xrightarrow{\text{canon}} {}^*Y \otimes {}^*X \boxtimes X \otimes Y = {}^*(X \otimes Y) \boxtimes (X \otimes Y) \leq T^R(\mathbb{1})$$