Recap

- Can model classical field theories by (0|1)-symplectic stacks $M$
- Procedure for building new theories from old "twisting". Assume $M$ formal over a base $B$. For us $B = \text{Bun}_G(X)$.
- Depends in particular on an odd symmetry $Q$, $Q^2 = 0$. Can find such a $Q$ from an action of supersymmetry algebra.
Example

No super Yang-Mills is a theory with an action of

\[ (so(4) \times \mathfrak{c}_6) \times \prod (s_\omega \omega W \ast s_\omega \omega W) \]

Choose \( Q_{\omega \omega} \) in \( s_\omega \omega W \) \( Q_{\omega \omega} \neq 0 \)

then Theorem (E-700)

the twist with \( Q_{\omega \omega} \) is equal to

\[ T^{[\omega]} \text{Higgs}_G (x) \text{ holomorphically twisted } N=0 \]

\[ B_{\omega \omega} (CP^{3|4}) \mapsto CP^{3|4}, BG \]
$M_{w!}(x) = T[1] \text{Maps}(T[x], B G)$

shifted tangent complex to $M_{w!}(x)$ at a $G$-bundle $P$ on $x$

$$\left( -\Omega_x^{x!, x} (x : f_P) \oplus \Omega_{x!}^{x!} (x : f_P) \right)_{[2]}$$

Two deformations we can consider:

1) Can turn on 2) on the two factors

2) Can turn on an iso between the two factors
For general $X$

deform $T C_{\lambda} X$ to $X \times \Delta$

Family over $A'$

Fiber over $a$ is $T C_{\lambda} X$

Fiber over $0$ is $X_{a}$

(Fiber over $\lambda$ is $X \times \Delta$)

Ringed spaces $(X, (\mathcal{O}(X, \lambda), \lambda))$

Deform $0 \mapsto \lambda$ to $\overline{\lambda} + \lambda$
Theorem (E-Yoo)

There's a family of classical field theory
over $\mathbb{C}^2$ whose fiber over $(\lambda, \mu)$
is $\text{Maps}(X^{Tw}, BG)$. This
coincides with the family of twists of
$(\text{N=4 SYM} / \Omega_{\infty}(x))$ defined by Kapustin-Witten.

At $(1,0)$ get $T:\text{Maps}(X^{Tw}, BG) = \mathcal{T} \mathcal{L} \mathcal{F} \text{lat}_G(X)$
at $(0,1)$ get $\text{Higgs}_G(X)_{\text{TR}}$. 

Local B"ohrner theory assigns to a compact submanifold $Y$ of $X$ the space $T^*[k-1]F_{\text{lat}}_G(Y)$, $k = \text{codim} Y$.

Solutions of differential equations on $Y$:

\[ \Omega_c(Y, g_p)_{[1]} \oplus \Omega_c(Y, g_p)_{[2]} \]

$X) \sim B_{\text{lust}}$

$A_{\text{lust}}$
In particular, if \( Y = pt \)

\[
\mathcal{M}_B (pt) \cong T^* [3] \mathcal{B} \mathcal{G}
\]

\( T \) - torsor

\[
\frac{g \cdot [2]}{\epsilon} \text{ - adjoint quotient}
\]

Definition: Local observables are functions on local solutions to equations of motion.

Here, \( \text{Obs}^{cl} \mathcal{B} \cong O(\frac{g \cdot [2]}{\epsilon}) \cong O(\hbar^{[2]} \ell_s \epsilon) \)
What structure does this have?

In general, in topological field theories of dim n (e.g. twists with topological supercharges), linear observables have the structure of a \( P^n \)-algebra — cdga \( \mathcal{A} \) with graded Lie bracket

\[ A \otimes A \rightarrow A[1-n] \]

which is a graded deformation for the product

\[ \mathcal{C}_X \subset \log X \quad \text{(e.g. manifold)} \]

\( \mathcal{O}_X \) is always a \( P^n \)-algebra

Here \( n=3 \)

\[ \mathcal{C}^3 \omega \cong \text{TM}^{3,\mathbb{R}} \quad \text{R-sym} \quad \text{so its functors form a Po-algebra.} \]

In fact this bracket is trivial for degrees zero...
Quantization \( H \xrightarrow{\pi} \)
In general, in this field theoretic top context
quantization of local observables are in particular
\( E_n \) deformations of the \( P_0 \)-algebra.
Are the non-local \( E_n \)-deformations of \( \mathcal{O}(\mathbb{R}^n/\mathbb{Z}_n) \)?
No. \( E_n \)-deformations are extended by \( \mathbb{R} \)-shifted Polyvector Fields (Toen)
\( \text{Pol}(\mathbb{R}^n/\mathbb{Z}_n, \mathbb{R}) \) are dg he algebra
Concentrated in even degree \( \Rightarrow \) no degree 1 els,
\( so \) subject to Maurer-Cartan.
Intro to Geometric Langlands

Idea: "Categorical non-abelian Fourier transform"

Roughly, $Bun_G(Σ) \cong$ smooth closed curve

Decompose $D$-modules on $Bun_G(Σ)$ in terms of nice "eigen-sheaves for natural operators"
The dual space (indexing eigenvalues) is $\text{Flat}_G(\mathfrak{g})$.

Langlands dual group

Conjecture ("Best hope")

There is an equivalence

$$D\text{-mod}(\text{Bun}_G(\mathfrak{g})) \simeq \text{Qcoh}(\text{Flat}_G(\mathfrak{g}))$$

"eigenstraves" $\rightarrow$ skyscraper & their eigenvalue
mnemonic: GoGoGo = G

Category of flat bundles = Category of vector bundles
in space of vector bundles = space of flat bundles

E.g., G abelian, Conjecture is a theorem (Laumon, Rothstein)
     it's literally a Fourier (-Mukai) transform

G non-abelian, Conjecture is false as written
     even if $\Sigma = \mathbb{P}^1$ (V. Lafforgue)
Conjecture (Arinkin–Gaitsgory)

$$D\text{-mod} (\text{Bun}_G(\mathcal{E})) \simeq \text{Ind Coh} \left( \text{Flat}_G(\mathcal{E}) \right)$$

KW claimed

category assigned to a curve $\mathcal{E}$

in $B$- Lust group $G$: $\mathcal{Q}_\text{coh}(\text{Loc}_G(\mathcal{E}))$

module of $G$-local systems

A Lust group $G$ $D\text{-mod} (\text{Bun}_G(\mathcal{E}))$

"nilpotent singular support"
Propose an ansatz.

If moduli space assigned to a curve $\Sigma$ is a shifted cotangent $T^*[\Sigma]$, say category or boundary conditions on $\Sigma$ is $\text{Ind}(\Sigma)$, then

"categorical geometric quantization".
This gives us

$$M_B(\Sigma) \cong \mathcal{T} \cdot \mathcal{L}_I \text{Flat}_G(\Sigma)$$

\Rightarrow \text{Cat of BCs} = \text{Ind Coh}(\text{Flat}_G(\Sigma))

Can also check

$$M_A(\Sigma) \cong \mathcal{T} \cdot \mathcal{L}_I \text{Bun}_G(\Sigma)_{dr}$$

\Rightarrow \text{Cat of BCs} = \text{Ind Coh}(\text{Bun}_G(\Sigma)_{dr}) = \text{D-mod}(\text{Bun}_G(\Sigma))$$