Gabor Frames

Karlheinz Gröchenig

Faculty of Mathematics, University of Vienna

http://homepage.univie.ac.at/karlheinz.groechenig/

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Outline

Time-Frequency Shifts

Translation operator: $T_x f(t) = f(t-x)$

Modulation operator $M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t)$

Time-frequency shift (phase-space shift): $z = (x, \xi) \in \mathbb{R}^{2d}$, $t \in \mathbb{R}^d$

$$\pi(z) f(t) = \underbrace{e^{2\pi i \xi \cdot t}}_{M_\xi} \underbrace{f(t-x)}_{T_x f\ (t)}$$

$\pi(z)$ is unitary on $L^2(\mathbb{R}^d)$ and an isometry on $L^p(\mathbb{R}^d)$

Gabor Systems

Fix lattice $\Lambda = A\mathbb{Z}^{2d}$ for $2d \times 2d$-matrix A with $\det A \neq 0$,
Fix "window" $g \in L^2(\mathbb{R}^d)$, $g \neq 0$

$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ Gabor system

Rectangular lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$
Separable lattice $\Lambda = P\mathbb{Z}^d \times Q\mathbb{Z}^d$, $P, Q \in \mathrm{GL}\,(d, \mathbb{R})$

Gabor Frames

$\mathcal{G}(g,\Lambda)$ is a Gabor frame, if for some $A, B > 0$

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

Equivalently, the frame operator

$$Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

is invertible on $L^2(\mathbb{R}^d)$, since

$$A\|f\|_2^2 = \langle Sf, f\rangle = \sum_{\lambda} \langle f, \pi(\lambda)g\rangle\langle \pi(\lambda)g, f\rangle \leq B\|f\|_2^2$$

Gabor Riesz Sequences

$\mathcal{G}(g,\Lambda)$ is a (Gabor) Riesz sequence, if for some $A, B > 0$

$$A\|\mathbf{c}\|_2^2 \leq \|\sum_{\lambda\in\Lambda} |c_\lambda \pi(\lambda) g\|_2^2 \leq B\|\mathbf{c}\|_2^2 \qquad \forall \mathbf{c} \in \ell^2(\Lambda)$$

Equivalently, the Gramian

$$(G\mathbf{c})_\lambda = \sum_{\mu\in\Lambda} \langle \pi(\mu) g, \pi(\lambda) g\rangle c_\mu$$

is invertible on $\ell^2(\Lambda)$, since

$$A\|\mathbf{c}\|_2^2 \leq \|\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g\|_2^2 = \langle G\mathbf{c}, \mathbf{c}\rangle \leq B\|\mathbf{c}\|_2^2$$

Gabor Expansions

Lemma

If $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame, then there exists a $\gamma \in L^2(\mathbb{R}^d)$, e.g., $\gamma = S^{-1}g$, such that

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)g$$

with unconditional convergence of the series in $L^2(\mathbb{R}^d)$.

Proof:

$$S\pi(\lambda) = \pi(\lambda)S \qquad \forall \lambda \in \Lambda$$

$$f = S^{-1}Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda){\color{red}S^{-1}}g$$

Riesz Sequences

Assume that $\mathcal{G}(g, \Lambda)$ is a Riesz sequence.
Transmit signal $f = \sum_{\mu \in \Lambda} c_\mu \pi(\lambda) g$.
At receiver compute correlations

$$y_\lambda = \langle f, \pi(\lambda) g \rangle = \sum_{\mu \in \Lambda} c_\mu \langle \pi(\mu) g, \pi(\lambda) g \rangle = (G\mathbf{c})_\lambda$$

$$\boxed{\text{so } \mathbf{y} = G\mathbf{c}.}$$

Consequently

$$\mathbf{c} = G^{-1} G \mathbf{c} = G^{-1} \mathbf{y}$$

Mathematical Problems

- Find conditions on g and Λ, such that $\mathcal{G}(g, \Lambda)$ is a frame.
- Find characterizations of Gabor frames
- Find (classes of) examples
- Given g, characterize all lattices Λ, such that $\mathcal{G}(g, \Lambda)$ is a frame.
- Relevance and Relations to other fields?

Gabor analysis

Duality of Gabor Systems

Definition: Let $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. If $\Lambda = A\mathbb{Z}^{2d}$ is a lattice, the lattice $\Lambda^\circ = \mathcal{J}(A^T)^{-1}\mathbb{Z}^{2d}$ is called the *adjoint* lattice.

Theorem (Janssen, Ron-Shen, Feichtinger-Kozek)

Let $g \in L^2(\mathbb{R}^d), g \neq 0$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. TFAE:

(i) *$\mathcal{G}(g, \Lambda)$ is a frame.*

(ii) *$\mathcal{G}(g, \Lambda^\circ)$ is a Riesz sequence.*

(iii) *There exists a dual window $\gamma \in L^2(\mathbb{R}^d)$, such that $\mathcal{G}(\gamma, \Lambda^\circ)$ is Bessel and γ satisfies the biorthogonality condition*

$$(\mathrm{vol}(\Lambda))^{-1}\langle \gamma, \pi(\mu)g\rangle = \delta_{\mu,0} \qquad \forall \mu \in \Lambda^\circ .$$

Duality II

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

if and only if

$$A'\|\mathbf{c}\|_2^2 \leq \|\sum_{\mu \in \Lambda^\circ} c_\mu \pi(\mu)g\|_2^2 \leq B'\|\mathbf{c}\|_2^2 \qquad \forall \mathbf{c} \in \ell^2(\Lambda^\circ)$$

Keywords for proof: gymnastics of time-frequency shifts, orthogonality relations for short-time Fourier transform, Poisson summation formula applied to spectrogram.

Poisson Summation Formula

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k) \qquad \text{for } f \in \mathcal{S}(\mathbb{R}^d)/M^1(\mathbb{R}^d)\,.$$

Note: If $h(z) = f(Az)$, then $\hat{h}(\zeta) = |\det A|^{-1}\hat{f}((A^T)^{-1}\zeta)$.

Characterization of Gabor Frames for Rectangular Lattices

Lemma

Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$. TFAE:

(i) $\mathcal{G}(g, \alpha, \beta)$ *is a frame.*

(ii) *There exist $A, B > 0$, such that for all $\mathbf{c} \in \ell^2(\mathbb{Z}^d)$ and almost all $x \in \mathbb{R}^d$*

$$A\|\mathbf{c}\|_2^2 \leq \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k g(x + \alpha j - \tfrac{k}{\beta} \Big|^2 \leq B\|\mathbf{c}\|_2^2 \,.$$

Frame Set

Definition

Given $g \in L^2(\mathbb{R}^d)$ fixed. Then

$$\mathcal{F}_{\text{full}}(g) = \{\Lambda \text{ lattice} : \mathcal{G}(g, \Lambda) \text{ is frame}\}$$

is called the full frame set of g, and

$$\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}^2_+ : \mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d) \text{ is frame}\}$$

is called the reduced frame set of g

Likewise

$$\mathcal{R}_{\text{full}}(g) = \{\Lambda \text{ lattice} : \mathcal{G}(g, \Lambda) \text{ is Riesz sequence }\}$$

and

$$\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}^2_+ : \mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d) \text{ is Riesz sequence }\}$$

Modulation Spaces

A function g belongs to the modulation space $M^1(\mathbb{R}^d)$ (Feichtinger's algebra), if

$$\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g\rangle| \, dz < \infty$$

or equivalently, if for fixed $\psi \in \mathcal{S}(\mathbb{R}^d), \psi \neq 0$

$$\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)\psi\rangle| \, dz < \infty$$

Lemma

For $f \in M^1(\mathbb{R}^d)$ the Poisson summation formula is valid.

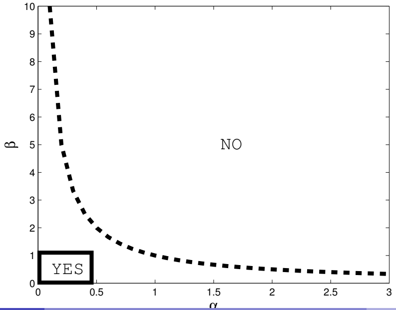
$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k) \qquad f \in M^1 .$$

Coarse Structure — Main Theorem

Theorem

Assume that $g \in M^1(\mathbb{R}^d)$. Then $\mathcal{F}_{\mathrm{full}}(g)$ is an open subset of $\{\Lambda \text{ lattice} : \mathrm{vol}\,(\Lambda) < 1\}$ and contains a neighborhood of $\mathbf{0}$.

Likewise, $\mathcal{F}(g)$ is an open subset of $\{(\alpha, \beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$ and contains a neighborhood of $(0, 0)$.



Examples/Questions

- Let $g(t) = te^{-\pi t^2}$ (first Hermite function)

Is $\mathcal{G}(g, \frac{2}{3}\mathbb{Z} \times \mathbb{Z})$ a frame?
Is $\mathcal{G}(g, 0.666666\mathbb{Z} \times \mathbb{Z})$ a frame?
[Are the points $(2/3, 1)$ and $(0.66666, 1)$ in $\mathcal{F}(g)$?]

- Let $g(t) = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]} = (1 - |x|)_+$.

Is $\mathcal{G}(g, \frac{2}{3}\mathbb{Z} \times \mathbb{Z})$ a frame?
Is $\mathcal{G}(g, \frac{1}{7}\mathbb{Z} \times \mathbb{Z})$ a frame?
Is $\mathcal{G}(g, \frac{1}{7}\mathbb{Z} \times 2.0001\mathbb{Z})$ a frame?
[Are $(2/3, 1), (1/7, 2), (1/7, 2.0001) \in \mathcal{F}(g)$?]

??

Precise Results about Gabor Frames in $1-D$

1. Lyubarski-Seip for Gaussian $g(t) = e^{-at^2}$
   $\mathcal{G}(g, \Lambda)$ is frame $\Leftrightarrow \mathrm{vol}(\Lambda) < 1$
2. Janssen-Strohmer for hyperbolic cosine $g(t) = (\cosh at)^{-1}$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow \alpha\beta < 1$
3. Janssen for exponential $g(t) = e^{-a|t|}$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow \alpha\beta < 1$
4. Janssen for one-sided exponential function $g(t) = e^{-at}\chi_{\mathbb{R}^+}(t)$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow \alpha\beta \leq 1$
5. $g(t) = (1 + at^2)^{-1}$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow \alpha\beta < 1$
6. $g(t) = (1 - iat)^{-1}$ for $a > 0$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow \alpha\beta \leq 1$

Weierstrass Sigma Function

Interpolation problem $G(\mu) = \delta_{\mu,0}$ is solved by

$$\sigma_{\Lambda^\circ}(z) = \prod_{\mu \in \Lambda^\circ, \mu \neq 0} \left(1 - \frac{z}{\mu}\right) e^{\frac{z}{\mu} + \frac{z^2}{2\mu^2}}$$

Weierstrass sigma-function

Classical result: $\Lambda = A\mathbb{Z}^{2d}$

Theorem

For suitable $a \in \mathbb{C}$, $|\sigma_{\Lambda^\circ}(z) e^{az^2}| = \mathcal{O}(e^{\pi s(\Lambda)|z|^2/2})$.

Corollary

(a) Interpolation problem $G(\mu) = \delta_{\mu,0}, \mu \in \Lambda^\circ$ possesses a solution $G \in \mathcal{F}$, if and only if $s(\Lambda) = |\det A| < 1$.
(b) $\mathcal{G}(\varphi, \Lambda)$ is a frame if and only if $s(\Lambda) < 1$ (Lyubarski, Seip '90).

Totally Positive Functions

Definition: A non-zero function g is totally positive, if for $x_1 < x_2 < \cdots < x_N$ and $y_1 < y_2 < \cdots < y_N$,$N \in \mathbb{N}$

$$\det \left[g(x_i - y_j)\right]_{1 \leq i,j \leq n} \geq 0$$

Examples: $g(t) = e^{-at^2}, (e^{at} + e^{-at})^{-1}, e^{-a|t|}, e^{-at}\chi_{\mathbb{R}^+}(t)$ AHA

Schoenberg: $g : \mathbb{R} \to \mathbb{R}$ *is totally positive, if and only if*

$$\hat{f}(\xi) = Ce^{-\gamma\xi^2 + 2\pi i\delta\xi} \prod_{\nu=1}^{\infty} (1 + 2\pi i \delta_\nu \xi)^{-1} e^{-2\pi i \delta_\nu \xi},$$

with real parameters $C, \gamma, \delta, \delta_\nu$ *satisfying*

$$C > 0, \quad \gamma \geq 0, \quad 0 < \gamma + \sum_{\nu=1}^{\infty} \delta_\nu^2 < \infty.$$

Gabor Frames and Totally Positive Functions

Definition: g is totally positive of finite type M, if

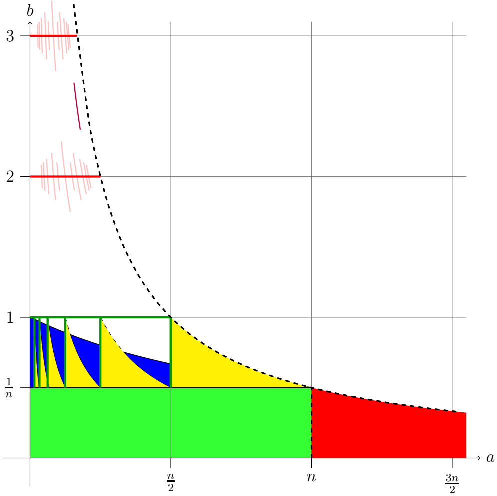
$$\hat{f}(\xi) = C \prod_{\nu=1}^{M} (1 + 2\pi i \delta_\nu \xi)^{-1} e^{-2\pi i \delta_\nu \xi},$$

Theorem (KG, J. Stöckler)

Assume that g is a totally positive function of finite type $M \geq 2$. Then $\mathcal{G}(g, \Lambda)$ is a frame, if and only if $\alpha\beta < 1$.

Splines

$g = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($n+1$-times)



Hermite Functions

$$h_n = c_n e^{\pi x^2} \frac{d^n}{dx^n}(e^{-2\pi x^2})$$

Theorem (KG, Lyubarski)

If $\mathrm{vol}(\Lambda) < \frac{1}{n+1}$*, then* $\mathcal{G}(h_n, \Lambda)$ *is a frame.*

However

Proposition (Lyubarski, Nes)

If $g \in L^2(\mathbb{R})$ *is odd and* $\alpha\beta = 1 - \frac{1}{N}$ *for* $N = 2, 3, \dots,$ *then* $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ *is NOT a frame.*

Hermite Functions

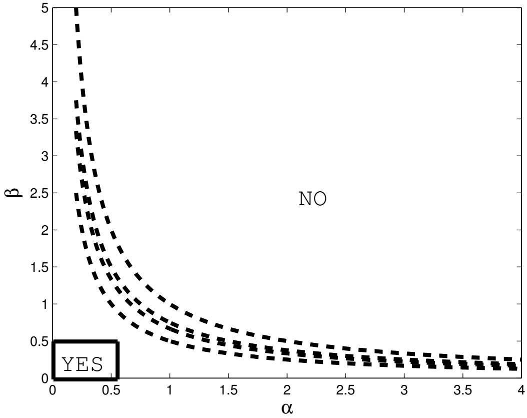


Figure: Possible frame set of odd function

Frame Bounds

Estimates for frame bounds for $\Lambda = \alpha\mathbb{Z}^2$

$$A(\alpha)\|f\|_2^2 \leq \sum_{\lambda\in\alpha\mathbb{Z}^2} |\langle f, \pi(\lambda)\varphi\rangle|^2 \leq B(\alpha)\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

Theorem (Borichev, KG, Lyubarski.)

For $1/2 \leq \alpha < 1$

$$\begin{aligned} c &\leq B(\alpha) \leq C \\ c(1-\alpha^2) &\leq A(\alpha) \leq C(1-\alpha^2) \end{aligned}$$

- Can be extended to other windows.

Frame Bounds II

Estimates for frame bounds for $\Lambda = \alpha\mathbb{Z}^2$
Let $\phi(t) = e^{-\pi t^2}$ and $A(\Lambda) = \|S^{-1}\|^{-1}$ and $B(\Lambda) = \|S\|$ be the optimal frame bounds in

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

Conjecture (Strohmer 2001):
(i) Among all rectangular lattices with $\alpha\beta = \sigma < 1$, the condition number $B(\Lambda)/A(\Lambda)$ is minimized by the square lattice $\sqrt{\sigma}\mathbb{Z}^2$.
(Proved by Faulhuber/Steinerberger for $\sigma = N^{-1}, N = 2, 3, \dots$)
(ii) Among *all* lattices with $\mathrm{vol}(\Lambda) = \sigma < 1$, the condition number $B(\Lambda)/A(\Lambda)$ is minimized by the hexagonal lattice. (open)

Further Directions

- Zak transform methods
- Gabor frames and function spaces (characterizations of modulation spaces)
- Gabor frames and pseudodifferential operators (almost diagonalization of pseudodifferential operators with Gabor frames)
- Gabor frames on finite Abelian groups
- Deformation results
- Gabor frames and Schrödinger equation
- ...

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Links

- My homepage:

http://homepage.univie.ac.at/karlheinz.groechenig/

- Hans Feichtinger's homepage:

http://www.univie.ac.at/nuhag-php/home/fei.php

- Numerical Harmonic Analysis Group:

www.nuhag.eu <http://www.nuhag.eu>
http://www.univie.ac.at/nuhag-php/bibtex/index.php (contains most/all papers related to Gabor Analysis)

- The Large Time-Frequency Analysis Toolbox

http://ltfat.sourceforge.net/