Motives, motivic Galois groups and periods
(Extended Abstract)

The notion of "motive" was invented by A. Grothendieck to serve as a universal cohomology theory for algebraic varieties. Roughly speaking, given an algebraic variety X over a field k there should exist cohomological invariants $H^i_{\mathcal{M}}(X, \mathbb{Q})$, non zero only for $0 \leq i \leq 2\dim(X)$, called *motives*, that determines the classical cohomological invariants such as the mixed Hodge structures on the singular cohomology $H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q})$ (for every choice of an embedding $k \hookrightarrow \mathbb{C}$) and the Galois representations on the ℓ-adic cohomology $H^i_{\text{ét}}(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ (for every choice of an algebraic closure $\overline{k}/k$). Moreover, these motives $H^i_{\mathcal{M}}(X, \mathbb{Q})$ should be algebraically defined and completely determined by the geometry of algebraic cycles (or, equivalently, by K-theory). Finally, the motives associated to X should govern the transcendence properties of the periods attached to X.

The above picture is of course largely conjectural. However, there have been some partial advances in the past years and the goal of the mini-course is to report on some of these advances. We hope to cover the following topics.

0.1. Motives à la Voevodsky.

Thanks to the work of Voevodsky, Morel–Voevodsky and others, we now have a very satisfactory construction of the *triangulated* category of motives. Unfortunately, this is much weaker than having an *Abelian* category of motives as demanded by Grothendieck,[1] but at least the relation with algebraic cycles and K-theory is very strong and as expected. (We recommend the introduction of [8] for a summary of the relations between motives, algebraic cycles and K-theory.)

Let k be a base field. (Our main interest is when the field k has characteristic zero, but the constructions make sense over general base-schemes.) Voevodsky's triangulated category of motives $\mathbf{DM}(k, \mathbb{Q})$ is constructed in a four-steps "enlargement" of the category Sm/k of smooth k-varieties:

(1) One considers the category of finite correspondences $\mathbf{SmCor}(k)$ from [8, Chapter 2]. The objects of this category are smooth k-varieties while the group of morphisms between X and Y is the group $\mathbf{Cor}(X, Y)$ of finite correspondences. (When k has characteristic zero, finite correspondences are exactly the elements of the group completion of the monoid of multivalued morphisms; see the introduction of the beautiful paper [17].)

(2) One considers the category $\mathbf{Str}(\mathrm{Sm}/k, \mathbb{Q})$ of étale sheaves with transfers from [15, Lecture 6]. These are contravariant functors from $\mathbf{SmCor}(k)$ to the category of $\mathbb{Q}$-vector spaces which are étale sheaves when restricted to Sm/k. A typical object of the category is $\mathbb{Q}_{\text{tr}}(X)$, for $X \in \mathrm{Sm}/k$, whose values on U is $\mathbf{Cor}(U, X) \otimes \mathbb{Q}$.

(3) The category $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ is the Verdier localization of the derived category $\mathbf{Str}(\mathrm{Sm}/k, \mathbb{Q})$ with respect to the smallest triangulated subcategory closed

[1] For instance, one drawback of this is that we don't know how to define the individual $H^i_{\mathcal{M}}(X, \mathbb{Q})$'s but we have a "motivic complex" that "contains" all the $H^i_{\mathcal{M}}(X, \mathbb{Q})$'s built together in a way that we don't yet understand.

under infinite sums and containing the complexes of the forms

$$[\mathbb{Q}_{\mathrm{tr}}(X) \stackrel{s_0}{\to} \mathbb{Q}_{\mathrm{tr}}(\mathbb{A}^1 \times X)].$$

This construction appears essentially in [15, Lecture 9].[2]

(4) The category $\mathbf{DM}(k,\mathbb{Q})$ is obtained from $\mathbf{DM}^{\mathrm{eff}}(k,\mathbb{Q})$ by inverting the Tate motive $T = \mathbb{Q}_{\mathrm{tr}}(\mathbb{P}^1,\infty)[-2]$ with respect to the tensor product. The "naive" way of doing this is explained in [8, page 192].[3] To do this correctedly, one appeals to the theory of symmetric spectra (see [11]) borrowed from stable homotopy theory, but we will probably pass over the technical details as they are irrelevant to our purposes. In any case, there is a natural functor

$$\Sigma^\infty_T : \mathbf{DM}^{\mathrm{eff}}(k,\mathbb{Q}) \to \mathbf{DM}(k,\mathbb{Q})$$

which is fully faithful. (Full faithfulness is a deep theorem of Voevodsky whose proof can be found in [8] !) The image of $\mathbb{Q}_{\mathrm{tr}}(X)$ by this functor is denoted by $\mathrm{M}(X)$; this is the *motive* attached to X.

Remark. — It is quite remarkable that the above construction can be *simplified* without changing the outcome ! Indeed, one can forget about transfers and consider simply the category $\mathbf{Shv}(\mathrm{Sm}/k,\mathbb{Q})$ of étale sheaves. Then, a Verdier localization of the derived category $\mathbf{D}(\mathbf{Shv}(\mathrm{Sm}/k,\mathbb{Q}))$ as in (3) gives the category $\mathbf{DA}^{\mathrm{eff},\mathrm{\acute{e}t}}(k,\mathbb{Q})$ which is equivalent to $\mathbf{DM}^{\mathrm{eff}}(k,\mathbb{Q})$ by a theorem of Morel and Cisinksi–Déglise. (See [3, Appendice B] for a simplified proof.) Also, inverting the Tate motive gives the category $\mathbf{DA}^{\mathrm{\acute{e}t}}(k,\mathbb{Q})$ which is equivalent to $\mathbf{DM}(k,\mathbb{Q})$. Although for the purpose of the mini-course there will be no good reasons for not using the simpler construction, we will stick to the categories $\mathbf{DM}^{\mathrm{eff}}(k,\mathbb{Q})$ and $\mathbf{DM}(k,\mathbb{Q})$.[4]

0.2. Motivic Galois groups.

The construction of $\mathbf{DM}(k,\mathbb{Q})$ explained above can be repeated in other contexts. For instance, one can replace the category Sm/k by the category CpVar of complex analytic smooth varieties. We denote by $\mathbf{AnDM}(\mathbb{Q})$ the triangulated category obtained in this way. It is an easy exercise to show that the category $\mathbf{AnDM}(\mathbb{Q})$ is canonically equivalent to the derived category of $\mathbb{Q}$-vector spaces $\mathbf{D}(\mathbb{Q})$. (See [3, §2.1] for a proof for the $\mathbf{DA}$-version.)

On the other hand, given an embedding $k \hookrightarrow \mathbb{C}$, the functor $\mathrm{Sm}/k \to \mathrm{CpVar}$, $X \mapsto X(\mathbb{C})$, induces an "obvious" functor $\mathrm{An}^* : \mathbf{DM}(k,\mathbb{Q}) \to \mathbf{AnDM}(\mathbb{Q})$. The

[2] In fact, it also appear in [8] but there the emphasis is on the Nisnevich topology. However, it is an easy exercise to check that with rational coefficients, the étale and Nisnevich topologies yield the same notion of sheaves with transfers.

[3] Note that the "naive" way yield the correct category when restricted to *geometric motives.*

[4] In fact, depending on the situation, one can prefer the **DM**-construction or the **DA**-construction:

- Over a perfect field, the **DM**-construction is very useful thanks to the Voevodsky's theory of "homotopy invariant presheaves with transfers" yielding an explicit model for the $\mathbb{A}^1$-localization functor. Such a model is not available for the **DA**-construction.
- The categories $\mathbf{DA}^{\mathrm{\acute{e}t}}(S,\Lambda)$ over general basis and for general coefficients rings have been studied extensively in [1, 2, 6] and their general functoriality is completely understood. This is not the case for the categories $\mathbf{DM}(S,\Lambda)$ for which the *localization axiom* is not known to hold in enough generality.

Betti realization can be defined as the composition of

$$\mathrm{B}^* : \mathbf{DM}(k, \mathbb{Q}) \xrightarrow{\mathrm{An}^*} \mathbf{AnDM}(\mathbb{Q}) \simeq \mathbf{D}(\mathbb{Q}).$$

The next ingredient for the construction of motivic Galois groups is the so-called *weak Tannakian formalism* developed in [3, §1]. Namely, given a monoidal functor $f : \mathcal{M} \to \mathcal{E}$ (e.g., $f = \mathrm{B}^*$) satisfying the following properties (see [3, Hypothèse 1.40])

(a) f admits a monoidal section e;
(b) f and e admit right adjoints g and c respectively;
(c) the natural morphism $g(A') \otimes B \to g(A' \otimes f(B))$ is an isomorphism for $A' \in \mathcal{E}$ and $B \in \mathcal{M}$,

the objet $\mathrm{H} = fg(\mathbf{1}) \in \mathcal{E}$ (where $\mathbf{1}$ is the unit for the tensor product) is naturally a Hopf algebra in $\mathcal{E}$. (This is [3, Théorème 1.45] whose proof is given in [3, §1].) Moreover, there is a universal factorization of f as

$$\mathcal{M} \xrightarrow{\tilde{f}} \mathbf{coMod}(\mathrm{H}) \xrightarrow{\mathrm{oub}} \mathcal{E}.$$

It is not difficult to show that the Betti realization B^* satisfies the conditions (a)–(c) above. We denote $\mathcal{H}_{\mathrm{mot}}(k) = \mathrm{B}^*\mathrm{B}_*\mathbb{Q}$ (where B_* is the right adjoint to B^*). This is the motivic Hopf algebra of k.

Note that, by construction, $\mathcal{H}_{\mathrm{mot}}(k)$ is an objet of $\mathbf{D}(\mathbb{Q})$. However, one has:

Conjecture. — *The homology of $\mathcal{H}_{\mathrm{mot}}(k)$ vanish except in degree zero.*

The right half of this conjecture is known by [3, Corollaire 2.105]. More precisely, one has $\mathrm{H}_i(\mathcal{H}_{\mathrm{mot}}(k)) = 0$ for $i < 0$. This is sufficient to insure that $\mathbf{H}_{\mathrm{mot}}(k) := \mathrm{H}_0(\mathcal{H}_{\mathrm{mot}}(k))$ inherits the Hopf algebra structure of $\mathcal{H}_{\mathrm{mot}}(k)$. It is then possible to make the following:

Definition. — *The motivic Galois group of k (associated to the embedding $k \hookrightarrow \mathbb{C}$) is $\mathbf{G}_{\mathrm{mot}}(k) := \mathrm{Spec}(\mathbf{H}_{\mathrm{mot}}(k))$.*

Remark. — There is a different construction of the motivic Galois group of a field due to M. Nori [16] (see [14] for a published accout of the construction). Nori's approach is based on an ingenious construction of an Abelian category of mixed motives. Unfortunately, this Abelian category of motives has some serious disadvantages:

- its construction is not purely algebraic and one needs to fix a Weil cohomology theory (e.g., singular cohomology) as an input;
- there is no interpretation of *ext*-groups between Nori's motives in terms of algebraic cycles or K-theory.

Nevertheless, it can shown [9] that the motivic Galois group constructed above is isomorphic to Nori's motivic Galois group. However, we feel it is important and more satisfactory to have a construction of the motivic Galois group built-in in the framework of Voevodsky's motives.

0.3. **Periods.**

In $\mathbf{DM}^{\mathrm{eff}}(k; \mathbb{Q})$ there is a natural well-known objet, namely the complex of sheaves $\Omega^{\bullet}_{/k}$ whose values on a smooth k-variety X is the global sections of the de Rham complex $\Omega^{\bullet}_{X/k}(X)$. (In fact, it is not obvious that $\Omega^{\bullet}_{/k}$ is a complex of sheaves *with transfers* but this is indeed true by [13]; alternatively, one can view $\Omega^{\bullet}_{/k}$ as an objet of $\mathbf{DA}^{\text{eff,ét}}(k,\mathbb{Q})$ which we know to be the same as $\mathbf{DM}^{\text{eff}}(k;\mathbb{Q})$.) We also note that $\Omega^\bullet_{/k}$ can be enhanced to an objet of $\mathbf{DM}(k;\mathbb{Q})$ that we denote by $\boldsymbol{\Omega}_{/k}$. In any case, it is an easy exercise to show that

$$\hom_{\mathbf{DM}(k,\mathbb{Q})}(\mathrm{M}(X), \boldsymbol{\Omega}_{/k}[i]) \simeq H^i_{\mathrm{dR}}(X).$$

A similar formula holds for $\mathrm{B}_*\mathbb{Q}$:

$$\hom_{\mathbf{DM}(k,\mathbb{Q})}(\mathrm{M}(X), \mathrm{B}_*\mathbb{Q}[i]) \simeq H^i_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q}).$$

(Again, an easy exercise !) The Grothendieck comparison theorem between algebraic de Rham cohomology and singular cohomology [10] can now be restated as an isomorphism in $\mathbf{DM}(k,\mathbb{Q})$:

$$\boldsymbol{\Omega}_{/k} \otimes_k \mathbb{C} \simeq (\mathrm{B}_*\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Going back to the definition of the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k)$, one deduces the following facts:

(i) $\mathcal{P}(k) := \mathrm{B}^*\boldsymbol{\Omega}_{/k}$ is a comodule over the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k)$;

(ii) $\mathcal{P}(k) \otimes_k \mathbb{C}$ is isomorphic (as a comodule) to $\mathcal{H}_{\mathrm{mot}}(k) \otimes_{\mathbb{Q}} \mathbb{C}$. Said differently, $\mathcal{P}(k)$ is a torsor over $\mathcal{H}_{\mathrm{mot}}(k)$.

Also, note that the vanishing of $\mathrm{H}_i(\mathcal{H}_{\mathrm{mot}}(k))$, for $i < 0$, implies the vanishing of $\mathrm{H}_i(\mathcal{P}(k))$ for $i < 0$. (In fact, this is proved in the other way round !) As before, we set $\mathbf{P}(k) := \mathrm{H}_0(\mathcal{P}(k))$ and we let $\mathbf{T}(k) = \mathrm{Spec}(\mathbf{P}(k))$.

Definition. — $\mathbf{T}(k)$ *is a torsor over the motivic Galois group* $\mathbf{G}_{\mathrm{mot}}(k)$*; it is called the* torsor of periods.

The torsor $\mathbf{T}(k)$ is defined over k (i.e., is a k-scheme). The Grothendieck comparison theorem yields a $\mathbb{C}$-point $\mathrm{comp} \in \mathbf{T}(k)(\mathbb{C})$. The following is a famous conjecture of Grothendieck and Kontsevich–Zagier:

Conjecture: *If k is a number field, the evaluation map*

$$\mathrm{comp}^* : \mathbf{P}(k) = \mathcal{O}(\mathbf{T}(k)) \to \mathbb{C}$$

is injective.

If time permits, I will also give a very concrete formulation of the previous conjecture using algebraic functions on polydiscs and their partial derivatives (see the introduction of [7]).

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