Mathematics of Signal Processing

Report on the Trimester Program at the Hausdorff Institute for Mathematics

Organizers:
Massimo Fornasier, TU Munich, `massimo.fornasier@ma.tum.de`
Mauro Maggioni, Duke University, `mauro@math.duke.edu`
Holger Rauhut, RWTH Aachen University, `rauhut@mathc.rwth-aachen.de`
Thomas Strohmer, UC Davies, `strohmer@math.ucdavis.edu`

Proposed Dates: January 4 - April 22, 2016

1 Topic Outline

Mathematical signal processing is a key basis of today's technology. Mobile phones, digital cameras, medical imaging devices, radar systems, internet communication, audio systems, chemical/physical sensors/detectors are a few examples of modern devices, which require advanced processing of signals, images and data. The progress in engineering goes hand in hand together with the one in corresponding applied mathematical fields. Engineering provides new challenging problems and inspiration for mathematics, while mathematics leads to advances in engineering by providing sophisticated solutions to these problems.
The *Shannon-Nyquist sampling theory*, initiated in the 1940ies, is one of the starting points for these developments by setting the foundations for measuring/sensing analog signals, transforming them into digital representations and for (tele-)communication of information. Such developments in mathematics and engineering have culminated in the 1980ies in the theory of *wavelets*, which has lead to new compression, denoising, and other image processing methods and at the same time provided deep mathematical insights with implications for instance to function space and operator theory.
Realizing the possibility of sketching analog signals by means of significantly reduced amount of numerical information inspired the development of the powerful mathematical theory of *compressed sensing*, initiated in 2004. It predicts that the Shannon-Nyquist rate may be significantly overcome in the sense that compressible (sparse) signals can be recovered with efficient algorithms from what was previously believed to be highly incompletely linear measurements. This surprising discovery has lead to an explosion of research on the applications of compressive sensing in many engineering areas such as biomedical imaging, radar technology, and astronomical signal processing. But besides that it has also lead to new deep mathematical results by established links between harmonic analysis, random matrix theory, and convex optimization.

Digitalization has currently come to a turning point, where not only traditionally considered analog signals such as audio, images, video, are collected, but entire daily activities are measured in different forms. The amount of data acquired, stored and transmitted on a daily basis is increasing at a rapid pace, and the ability of efficiently processing such large data sets is becoming even more important presently. Together with the technological advances, we are facing an increasing demand of novel mathematical methods to perform efficient information processing at this scale. Real-life data analysis challenges require fundamentally new ideas and approaches, with significant consequences for potential cross-disciplinary mathematical developments. In fact, mathematical techniques involved in signal and data processing and proofs of corresponding theorems involve (and sometimes connect) various fields including harmonic analysis, optimization, probability theory, Banach space geometry, numerical linear algebra, graph theory, and more. The variety and the sophistication of the involved mathematics has made some of the challenging problems very attractive also to pure mathematicians.

The trimester program aimed at bringing together world experts as well as talented postdocs and PhD students working on current fields of mathematical signal processing. Having several outstanding researchers in the field together at one place for an extended period of time without many other obligations has been very fruitful for the further development of the field, for identifying new mathematical problems, exploring new ideas and directions, and providing solutions to the corresponding challenging mathematical problems. In addition to mathematicians, we also invited a small number of very mathematically oriented electrical engineers, computer scientists and physicists who provided input on the actual needs in signal processing applications. This direct interaction of mathematicians with engineers and computer scientists is central for this field to further breakthroughs and inspiration in the area of mathematical signal processing. The field is currently very active and quickly moving, and the percentage of young outstanding researchers in the community is very large so that we have been able to attract a significant number of highly talented postdocs and PhD students for this program.

2 Scientific Programme

Below we illustrate the core topics that have been at focus of the trimester. There have also been activities on a smaller scale in related other topics, but we will not mention all of them here. For each core topic we report some of the relevant events (talks, lecture series, and workshops etc.) and results collected in the preprints/publications of HIM guest scientists prepared during the trimester.

2.1 Sampling theory, Gabor frames, and functional analysis

Shannon sampling theory is one of the milestones of mathematical signal processing and forms the basis for current analog-to-digital signal conversion techniques as well as of

most of the present communication technology. Classical sampling theory addresses the problem of recovering a band-limited function from function values collected at the so-called Nyquist rate.
This classical methodology has been generalized significantly. This includes operator sampling, inspired by channel identification problems in wireless communication or more general inverse problems in system identification in engineering processes as well as the theory of transformation sampling including wavelet and Gabor (time-frequency) transformations. These developments have led to the nowadays well-established theory of redundant decompositions, known under the name of frames. The crucial problems in this context are the formulation of new sampling methods, i.e., the design of new frames, their efficiency in terms of the balance between acquired information, recovery accuracy, robustness with respect to perturbations and computational complexity. Practical usage of frames today includes robust coding and design as well as analysis of filter banks, but also many tasks for image processing such as denoising and inpainting. The theory of frames, as a modern view of classical sampling theory, has several and diverse connections with other branches of mathematics, for instance optimal sphere packing problems in high-dimensions, finite Banach space geometry, and also the rich theory of function spaces. Challenging problems in frame theory include the construction of specific frames for certain purposes and with certain properties. Another important task is the design and the analysis of new efficient signal processing algorithms based on frames. Frames are also used to characterize Banach spaces of functions, including modulation spaces, Besov spaces, shearlet spaces, in connection to abstract harmonic analysis and the theory of square-integrable representations of groups.

Among the relevant events related to this core topic we mention the lectures by Karlheinz Gröchenig on “Gabor Analysis and its Mysteries” during the Winter School “Advances in Mathematics of Signal Processing” (January 11-15, 2016), the lecture series of Hans Feichtinger on “Fourier Analysis via the Banach Gelfand Triple” and Franz Luef on “The finite Heisenberg group in noncommutative geometry”, the Workshops “Finite Weyl-Heisenberg Groups in mathematics, quantum physics, and engineering” (February 22-24, 2016), and “Harmonic Analysis, Graphs and Learning” (March 14-18, 2016), including the talks

- Ingemar Bengtsson: Weyl-Heisenberg groups in quantum mechanics
- Peter Jung: Some Aspects of Weyl-Heisenberg Signal Design in Wireless Communication
- Werner Kozek: Two selected applications of the Weyl-Heisenberg group in automotive engineering
- Romanos Malikiosis: Full spark Gabor frames in finite dimensions
- Stephan Dahlke: Shearlet Coorbit Spaces and Shearlet Groups

- Holger Boche: Banach-Steinhaus Theory Revisited: Lineability, Spaceability, and related Questions for Phase Retrieval Problems for Functions with finite Energy

In the field of sampling theory, Gabor frames, and functional analysis, we report a few relevant preprints/papers presented by HIM guest scientists during the trimester

- 2016a02 Aldroubi, A. ; Cabrelli, C.; Cakmak, A. F.; Molter, U.; Petrosyan, A.: Iterative actions of normal operators, arXiv:1602.04527, MR3579135
- 2016a05 Boche, Holger; Tampubolon, Ezra: Mathematics of signal design for communication systems, MR3497595
- 2016a07 Boche, Holger; Mnich, Ullrich J.: Spaceability for sets of bandlimited input functions and stable linear time-invariant systems with divergence behavior
- 2016a10 Alberti, Giovanni S.; Dahlke, Stephan; De Mari, Filippo; De Vito, Ernesto; Fhr, Hartmut: Recent progress in shearlet theory: systematic construction of shearlet dilation groups, characterization of wavefront sets, and new embeddings, arXiv:1605.02873
- 2016a14 Franklin, David J.; Hogan, Jeffrey A.; Larkin, Kieran G.: Hardy, Paley-Wiener and Bernstein spaces in Clifford analysis, MR3662507
- 2016a22 Juestel, Dominik: The Zak transform on strongly proper G-spaces and its applications, arXiv:1605.05168
- 2016a28 Ghaani Farashahi, Arash: Square-integrability of multivariate metaplectic wave-packet representations, MR3622579
- 2016a29 Ghaani Farashahi, Arash: Square-integrability of metaplectic wave-packet representations on L2(R), MR3595232
- 2016a34 Feichtinger, Hans Georg; Voigtlaender, Felix: From Frazier-Jawerth characterizations of Besov spaces to wavelets and decomposition spaces, arXiv:1606.04924
- 2016a40 Gröchenig, Karlheinz; Rottensteiner, David: Orthonormal bases in the orbit of square-integrable representations of nilpotent Lie groups, arXiv:1706.06034

2.2 Compressive sensing, quantization, and low-complexity models

A major shift of paradigm in the view of sampling theory occurred in the past ten years with the development of the theory of compressive sensing. It predicts the robust recovery of a compressible (sparse) signal from vastly incomplete linear random measurements via efficient methods such as convex optimization or certain greedy algorithms. One major difference with respect to the classical sampling theory, is that signal recovery is not anymore a linear process. This theory triggered huge research efforts, for instance, on the development of a large scope of new algorithms, tailored to different applications, and on the fine analysis of (Gaussian and structured) random measurements in this context.

The extension of the paradigm of compressed sensing to more general situations is the current subject of an enormous activity and new results and breakthroughs are appearing at a very fast pace. For instance, we refer to the recovery of low rank matrices from few linear random measurements. This has important applications in high-dimensional data analysis, but also forms the basis for the development of a robust theory of compressed sensing via nonlinear measurements. We mention the recent results in the celebrated phase retrieval problem, important in diffraction imaging and X-ray crystallography, showing that it is possible to recover (sparse and non-sparse) signals from the absolute values of their scalar products with respect to a small number of elements of certain finite frames. Again we witness a new explosion of research on efficient algorithmic methods for achieving robust recovery from nonadaptive compressed measurements in these more general contexts.

The mathematical analysis of (structured) random matrices requires sophisticated tools from this area, including methods from Banach space geometry and bounding random process via generic chaining. Compressed sensing has pushed new ground breaking results at the foundations of these fields, but the solution of some compressed sensing problems would require further progress on tools for bounding certain stochastic processes, for instance.

Despite the enormous impact of compressive sensing in terms of developments of new theoretical results, numerical methods and exciting applications, a number of problems remain to be solved. For instance, we mention the phenomenon of noise folding, that is, the amplification of the noise on the signal prior to measurement, which may greatly reduce the accuracy of the recovery, unless one resumes a number of measurements in the order of the classical Nyquist-rate. This crucial applied problem is currently a major subject of intensive research, which calls for new ideas and major breakthroughs. Also efficient software-hardware implementations of recovery from compressive sensing measurements are crucial issues concerning applicability. Moreover, extending the notion of low intrinsic complexity from sparse vectors and low rank matrices to other interesting structures (e.g. low rank tensors, combination of sparsity and low rank assumptions etc.), and developing corresponding algorithms and measurement schemes poses a lot of open questions. Furthermore, experts predict that compressive sensing and sparsity will play an important role in 5th-generation wireless communication systems (to be released around 2020). For this prediction to become reality, challenging mathematical problems at the interface of signal processing, convex optimization, harmonic analysis, and communications engineering still need to be solved.

The current research on quantization in mathematical signal processing addresses efficient methods for obtaining robust quantizers of analog signals, including low-bit quantizations such as Sigma-Delta (abbreviated by $\Sigma\Delta$) schemes. Also the use of quantization within the new paradigm of compressive sensing is a major research topic including one-bit compressed sensing, extensions to recovery of low rank matrices and analysis of structured random measurement schemes.

Among the relevant events related to this core topic we mention the lectures by Simon Foucart on "Essentials of Compressive Sensing" during the Winter School "Advances in Mathematics of Signal Processing" (January 11-15, 2016), the lecture series of Holger Rauhut on "Chaining and small ball estimates with applications to compressive sensing", the Workshop "Low Complexity Models in Signal Processing" (February 15-19, 2016) including the talks

- Laurent Jacques / Valerio Cambareri: Small width, low distortions: quantized random projections of low-complexity signal sets
- Xiadong Li: Phase Retrieval: from Convex to Nonconvex Methods
- Ke Wei: Low rank matrix recovery: From iterative hard thresholding to Riemannian optimization
- Andre Uschmajew: Finding a low-rank basis in a matrix subspace
- Bart Vandereycken: Manifold optimization for low rank matrix and tensor completion
- Sjoerd Dirksen: On the gap between restricted isometry properties and sparse recovery conditions
- Gabriel Peyre: Exact Support Recovery for Sparse Spikes Deconvolution
- Ben Adcock: Sensing and parallel acquisition
- Simone Brugiapaglia: CORSING: sparse approximation of PDEs based on compressed sensing
- Jelani Nelson: Optimal space and fast heavy hitters with high probability
- David Gross: Diamond norm as improved regularizer for low rank matrix recovery
- Felix Krahmer: Matrix factorization with binary components - uniqueness in a randomized model

In the field of compressive sensing, quantization and low-complexity models, we report a few relevant preprints/papers presented by HIM guest scientists during the trimester

- 2016a04 Rauhut, Holger; Schneider, Reinhold; Stojanac, Zeljka: Low rank tensor recovery via iterative hard thresholding, arXiv:1602.05217 MR3624675
- 2016a11 Kueng, Richard; Jung, Peter: Robust nonnegative sparse recovery and the nullspace property of 0/1 measurements, arXiv:1603.07997

- 2016a15 Bouchot, Jean-Luc; Rauhut, Holger; Schwab, Christoph: Multi-level compressed sensing Petrov-Galerkin discretization of high-dimensional parametric PDEs: arXiv:1701.01671
- 2016a16 Iwen, Mark A.; Preskitt, Brian; Saab, Rayan; Viswanathan, Aditya: Phase retrieval from local measurements: improved robustness via eigenvector-based angular synchronization, arXiv:1612.01182
- 2016a18 Kueng, Richard; Zhu, Huangjun; Gross, David: Low rank matrix recovery from Clifford orbits, arXiv:1610.08070
- 2016a21 Kliesch, Martin; Kng, Richard; Eisert, Jens; Gross, David: Improving compressed sensing with the diamond norm, arXiv:1511.01513, MR3599093
- 2016a30 Pfander, Götz E.; Salanevich, Palina: Robust phase retrieval algorithm for time-frequency structured measurements, arXiv:1611.02540
- 2016a31 Feng, Joe-Mei; Krahmer, Felix; Saab, Rayan: Quantized compressed sensing for partial random circulant matrices, arXiv:1702.04711
- 2016a32 Rauhut, Holger; Stojanac, Zeljka: Tensor theta norms and low rank recovery, arXiv:1505.05175
- 2016a33 Walk, Philipp; Jung, Peter; Pfander, Götz E.; Hassibi, Babak: Blind deconvolution with additional autocorrelations via convex programs, arXiv:1701.04890
- 2016a35 Kümmerle, Christian; Sigl, Juliane: Harmonic mean iteratively reweighted least squares for low-rank matrix recovery, arXiv:1703.05038
- 2016a37 Krahmer, Felix; Kruschel, Christian; Sandbichler: Michael Total variation minimization in compressed sensing, arXiv:1704.02105
- 2016a38 Jung, Peter; Krahmer, Felix; Stoeger, Dominik: Blind demixing and deconvolution at near-optimal rate, arXiv:1704.04178
- 2016a41 Mendelson, Shahar; Rauhut, Holger; Ward, Rachel: Improved bounds for sparse recovery from subsampled random convolutions, arXiv:1610.04983

2.3 High-dimensional data analysis and machine learning

Current developments in the field of machine learning vigorously led by the community of Computer Science are producing exceptional results on highly complex decision tasks. This great empirical success based on deep learning shows that algorithms can beat humans in several tasks, such as image classification or learning how to play complex games (chess, Go, etc.). Nevertheless the mathematical modeling of such algorithms is still in its infancy. Mathematical guarantees and understanding of the behavior of machine learning are crucial, however. Without such comprehension, practitioners are moving in the blind, operating solely based on empirical evidences without a proper theory to explain them. The mathematical community participating at the trimester included forerunners, who have been first to contribute novel mathematically founded machine learning methods, such as diffusion maps and kernel methods. More importantly, they gave first theoretical understanding and insights on deep learning methods. The trimester has clearly helped the mathematical community to interact on these novel topics and to make efforts of sheding light on the theoretical mysteries of machine learning. Once again we saw harmonic analysis tools, which already showed a great impact in the field, jointly with high-dimensional optimization, and probabilistic techniques to play a major role in the understanding of the new machine learning challenges.

Among the relevant events related to this core topic we mention the lectures by Francis Bach on "Large-scale Machine Learning and Convex Optimization" and Stephen Wright on "Fundamentals of Optimization in Signal Processing" during the Winter School "Advances in Mathematics of Signal Processing" (January 11-15, 2016), the lecture series of Suvrit Sra on "Aspects of Convex, Nonconvex, and Geometric Optimization", the Workshops "Low Complexity Models in Signal Processing" (February 15-19, 2016), and "Harmonic Analysis, Graphs and Learning" (March 14-18, 2016), including the talks

- Jan Vybiral: Non-asymptotic analysis of ℓ_1-support vector machines
- Lorenzo Rosasco: Less is more: optimal learning with stochastic projection regularization
- Albert Cohen: Data assimilation in reduced modeling
- Rachel Ward: Some recent results in hashing and clustering
- Gilad Lerman: Fast, Robust and Non-convex Subspace Recovery
- Andrea Montanari: Graph estimation via semi-definite programming
- Joan Bruna: Signal Recovery from Scattering Convolutional Networks
- Helmut Boelcskei: Deep convolutional feature extraction: Theory and new architectures
- Christine De Mol: Combining information or forecasts for predicting economic variables
- Philipp Grohs: Some mathematical properties of deep convolutional networks

In the field of mathematical analysis and high-dimensional optimization for machine learning, we report a few relevant preprints/papers presented by HIM guest scientists during the trimester

- 2016a01: Bongini, Mattia; Fornasier, Massimo; Hansen, Markus; Maggioni, Mauro: Inferring interaction rules from observations of evolutive systems I: the variational approach, arXiv:1602.00342, MathSciNet MR3636616

- 2016a04 Rauhut, Holger; Schneider, Reinhold; Stojanac, Zeljka: Low rank tensor recovery via iterative hard thresholding arXiv:1602.05217, MathSciNet MR3624675

- 2016a32 Rauhut, Holger; Stojanac, Zeljka: Tensor theta norms and low rank recovery, arXiv:1505.05175

- 2016a35 Kümmerle, Christian; Sigl, Juliane: Harmonic mean iteratively reweighted least squares for low-rank matrix recovery, arXiv:1703.05038

3 Organization of the program

There have 1-3 seminar talks by the participants every week. This includes also lecture series of about 3-4 lectures by a few senior participants, for instance the lecture series of Suvrit Sra on “Aspects of Convex, Nonconvex, and Geometric Optimization” and Holger Rauhut on “Chaining and small ball estimates with applications to compressive sensing”. Moreover, several discussions on open problems were held.

We organized four major focused activities:

- Winter School on “Advances in Mathematics of Signal Processing” (January 11-15)[1] (78 participants)

- Workshop on “Low Complexity Models in Signal Processing” (February 15-19)[2] (72 participants)

- Workshop on “Finite Weyl-Heisenberg Groups in mathematics”, quantum physics, and engineering (Feburary 22-24)[3] (ca. 53 participants)

- Workshop on “Harmonic Analysis, Graphs and Learning” (March 14-18) [4] (ca. 91 participants)

---

[1] https://www.him.uni-bonn.de/programs/past-programs/past-trimester-programs/signal-processing-2016/winter-school/

[2] https://www.him.uni-bonn.de/programs/past-programs/past-trimester-programs/signal-processing-2016/workshop-on-low-complexity-models/

[3] https://www.him.uni-bonn.de/programs/past-programs/past-trimester-programs/signal-processing-2016/workshop-on-finite-weyl-heisenberg-groups/

[4] https://www.him.uni-bonn.de/programs/past-programs/past-trimester-programs/signal-processing-2016/workshop-on-harmonic-analysis-graphs-and-learning/

4 Statistics

The trimester has been attended by 176 participants from 29 countries. Among these were 31 female participants. 55 PhD students and 33 PostDocs have attended. During the trimester 33 talks have been recorded and 40 preprints/publications have been registered.