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IVAN CHELTSOV, JIHUN PARK, CONSTANTIN SHRAMOV

During our study at Hausdorff Research Institute for Mathematics in Bonn for the period 1st August — 4th September 2012, we studied three closely related topics: Kähler–Einstein metrics on Fano orbifolds, birational automorphisms of Fano orbifolds, and non-rationality of Fano hypersurfaces. Let us describe the results we proved during our visit.

## 1. Kähler–Einstein metrics on Fano orbifolds

A central theme in differential geometry is to try and characterize a given geometric structure by a metric with best curvature properties. In general, a metric may not be characterized by curvature properties, since the number of degrees of freedom may not even match. However, in Kähler geometry, there is such a natural question: for a given compact complex Kähler manifold, determine when it admits a Kähler–Einstein metric (a Kähler metric whose Ricci curvature is proportional to the metric tensor). This problem, known as the Calabi problem, was solved by Yau and Aubin for complex manifolds with negative or vanishing first Chern class (see [1], [43], and cf. [44]). The problem of existence of Kahler-Einstein metrics on compact complex manifolds with positive first Chern class, i.e. Fano manifolds, is a very subtle problem that still remains unsolved (see [44], [26], [27], [28]). For two-dimensional Fano manifolds, i.e. for del Pezzo surfaces, the Calabi problem has been completely solved by Tian and Yau (see [40], [41]). For smooth toric Fano manifolds this problem has been completely solved by Wang and Zhu (see [42]). Note that Calabi problem makes sense only for complex manifolds with negative, vanishing, or positive first Chern class.

The existence of an orbifold Kahler–Einstein metric on a Fano orbifold $X$ is equivalent to the existence of a solution of a global complex Monge–Ampere equation on $X$. This problem remains out of reach even in dimension two. We know many obstructions to the existence of such orbifold metric (see [28]). However, we know only known sufficient condition for the existence of such metric, which can be formulated in terms of the so-called $\alpha$-invariant of Tian introduced in [39]. Recall that the $\alpha$-invariant of Tian of the Fano orbifold $X$ measures singularities of the divisors in $X$ that are given by the zeroes of global holomorphic sections of the line bundle $(\wedge^{\dim(X)}\mathcal{T}_X)^{\otimes n}$ for $n \gg 0$. It follows from the works of Tian, Nadel, Kollár and Demailly that the lower bound

$$\alpha(X) > \frac{\dim(X)}{\dim(X)+1}$$

would imply the existence of an orbifold Kähler–Einstein metric on the Fano orbifold $X$ (see [39], [34], [24], [16, Theorem A.3]).

During our stay at Hausdorff Research Institute for Mathematics in Bonn, we (Ivan Cheltsov and Jihun Park) proved

**Theorem 1** ([15])**.** Let $X$ be a general hypersurface of degree $n = 4$ or 5 in $\mathbb{P}^n$. Then

$$\alpha(X) \geqslant \begin{cases} \frac{7}{9} & \text{for } n = 4; \\ \\ \frac{5}{6} & \text{for } n = 5. \end{cases}$$

This result has a long history. It was proved in [3, Theorem 1.3] and [11, Theorem 3.3] that $\alpha$-invariant of *any smooth* hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 3$ is at least $\frac{n-1}{n}$ and this bound is sharp, i.e. for every $n$ there are smooth hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 3$ whose $\alpha$-invariant is exactly $\frac{n-1}{n}$. Moreover, it follows from [35, Theorem 2] that $\alpha$-invariant of *general* hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 6$ is just 1. For general cubic surface in $\mathbb{P}^3$, the $\alpha$-invariant is $3/4$ by [7, Theorem 1.7], and the existence of Kähler–Einstein metric on every smooth cubic surface is proved in [40]. Since $\mathbb{P}^1$ has Kähler–Einstein metric, i.e. the Fubini-Study metric, consequently, we obtain

**Corollary 2.** A general hypersurface of degree $n \geqslant 2$ in $\mathbb{P}^n$ has a Kähler–Einstein metric.

We started to prove Theorem 1 long time ago. In fact, we proved a slightly stronger result in [15]. However, some time ago we discovered a gap in the original proof of Theorem 1 (the proof of the quintic fourfold part was faulty). During our stay at Hausdorff Research Institute for Mathematics in Bonn, we removed this gap and finished the proof of Theorem 1.

## 2. Birational automorphism of Fano orbifolds

During last three years, we (Ivan Cheltsov and Shramov) worked a lot on the the conjugacy classes in $\mathrm{Cr}_3(\mathbb{C})$ of the subgroups $\mathrm{PSL}_2(\mathbb{F}_7)$ and $\mathrm{A}_6$. Namely, we proved that $\mathrm{Cr}_3(\mathbb{C})$ has at least 5 non-conjugate subgroups isomorphic to $\mathrm{A}_6$ (see [20]), and we proved that $\mathrm{Cr}_3(\mathbb{C})$ has at least 3 non-conjugate subgroups isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$ (see [19]). Our methods were based on the technique developed by us in [17] and [18], classical approach like in [25], and some recent local inequalities found in [22].

We planned to continue research in this direction at Hausdorff Research Institute for Mathematics in Bonn. However, Constantin Shramov was unable to come to Bonn because of visa problems (he applied to German visa too late because of his intensive travel in June and July 2012). Nevertheless, we still were able to work together using Skype for communication.

During our stay at Hausdorff Research Institute for Mathematics in Bonn, we (Ivan Cheltsov and Constantin Shramov) studied conjugacy classes of in $\mathrm{Cr}_3(\mathbb{C})$ of the subgroups isomorphic to $\mathrm{A}_5$, i.e. the the group of symmetries of the icosahedron. This problem is not easy, because $\mathrm{A}_5$ acts birationally both on $\mathbb{P}^1$ and $\mathbb{P}^2$, which implies that it has many different embeddings into $\mathrm{Cr}_3(\mathbb{C})$. In order to study them, we had to consider its biregular action on possibly singular Fano threefolds and study $\mathrm{A}_5$-equivariant birational transformations of these threefolds. Our main target was smooth Fano threeefold of index 2 and degree 5, which is usually denoted by $V_5$ (such threefold is unique).

**Example 3** ([32])**.** The Grassmaniann $G(2,5)$ of 2-dimensional planes is naturally embedded into the projective space $\mathbb{P}^9$. Let $X$ be a section of $G(2,5)$ by a general subspace of codimension 3. Then $V_5$ is a Fano threefold with $\mathrm{Pic}(V_5) \cong \mathbb{Z}[H]$ and $-K_{V_5} \sim 2H$, where $H$ is the class of a hyperplane section of $V_5 \subset \mathbb{P}^6$. Moreover, the threefold $V_5$ is rational and $\mathrm{Aut}(V_5) \cong \mathrm{PSL}_2(\mathbb{C})$. Thus, the threefold $V_5$ admits a faithful action of the group $\mathrm{A}_5$, since $\mathrm{PSL}_2(\mathbb{C})$ contains a unique subgroup isomorphic to $\mathrm{A}_5$.

This threefold has been studied by many people. We almost proved

**Theorem 4** ([21])**.** Let $V_5$ be the threefold constructed in Example 3. Then $V_5$ is $\mathrm{A}_5$-birationally rigid (see [20, Definition 1.9]), and $\mathrm{Bir}^{\mathrm{A}_5}(V_5)$ is isomorphic to $\mathrm{S}_5 \times \mathbb{Z}_2$.

We plan to finish the proof of this theorem in 2012. We proved 85% of Theorem 4.

## 3. Non-rationality of Fano hypersurfaces

In 1979 Reid announced the 95 families of $K3$ surfaces in three dimensional weighted projective spaces (see [36]). After this, Iano-Fletcher, who was a Ph.D. student of M. Ried, announced the 95 families of weighted Fano threefold hypersurfaces in his Ph.D. dissertation in

1988 (see [29]). These are quasi-smooth well-formed hypersurfaces in weighted projective spaces $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ of degrees $d = \sum_{i=0}^{4} a_i$ with only terminal singularities. It turns out that $a_0$ is always 1 if we assume that $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$, and the Reid's 95 familes of $K3$ surfaces are anticanonical sections of Fano threefolds found by Iano-Fletcher. Note that each family among these 95 ones are determined by quadruples of non-decreasing positive integers $(a_1, a_2, a_3, a_4)$. The completeness of the list of Reid and Iano-Fletcher was proved much later by Johnson and Kollár (see [33]).

In late nineties, these 95 families revived and attracted birational geometers to study their properties such as birational rigidity (for the definition of birational rigidity see [23], [22], [30], [4]), groups of birational automorphisms, elliptic fibration structures, and so forth (see, for instance, [2], [5], [12], [13], [23], [37], and [38]). In particular, the paper [23] proves that a general hypersurface in each of the 95 families of weighted Fano threefold hypersurfaces is birationally rigid. In other words, it cannot be birationally transformed into a Mori fibred spaces except itself, which generalizes the classical result of Iskovskikh and Manin about the non-rationality of every smooth quartic threefold (see [31]). The paper [12] studies elliptic fibration structures, Halphen pencils, and groups of birational automorphisms of general hypersurfaces of the 95 families. In this paper together with the paper [2], it is proven that a general hypersurface in the families No. 3, 60, 75, 83, 87, 93 (see [23] for the entry numbers) cannot have any elliptic fibration structures. Furthermore the paper [5] classifies all elliptic fibration structures on a general hypersurface in each of the 95 families of weighted Fano threefold hypersurfaces. In addition, the paper [13] classifies all Halphen pencils on a general hypersurface in each of the 95 families of weighted Fano threefold hypersurfaces, and the paper [10] does this for every smooth quartic threefold. The problem of existence of orbifold Kähler-Einstein metrics on the 95 families of weighted Fano threefold hypersurfaces was studied in [6], [8], [9] and [15] (see Theorem 2).

As mentioned above, Corti, Pukhlikov, and Reid proved that a *general* hypersurface in each of these 95 families of weighted Fano threefold hypersurfaces is birationally rigid (see [23]). However, they did not prove that *every quasi-smooth* hypersurface in each of these 95 families is birationally rigid, but they conjectured this (see [23, Main Theorem-Conjecture 1.3]). During our stay at Hausdorff Research Institute for Mathematics in Bonn, we (Ivan Cheltsov and Jihun Park) almost proved

**Theorem 5** ([14])**.** Every quasi-smooth hypersurface in the families of the 95 families of weighted Fano threefold hypersurfaces is birationally rigid and, in particular, non-rational.

We hope to finish by the end of 2012. We proved 95% of this Theorem 4.

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