Two boundary centralizer algebras for $\mathfrak{gl}(n|m)$

U. of Oklahoma

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Let $A$, $B$ be two algebras, $W$ be a $\mathbb{C}$-vector space. We study the centralizing actions

$$A \hookrightarrow W \bowtie B$$
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<thead>
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<th>$A$</th>
<th>$W$</th>
<th>$B$</th>
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<tbody>
<tr>
<td>Schur (1905)</td>
<td>$\mathfrak{gl}_n(\mathbb{C})$</td>
<td>$V^\otimes d$</td>
<td>sym. gp</td>
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<td>Arakawa-</td>
<td>$\mathfrak{sl}_n(\mathbb{C})$</td>
<td>$M \otimes V^\otimes d$</td>
<td>degenerate affine Hecke alg.</td>
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<td>Suzuki (1998)</td>
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<td>Daugherty</td>
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<td>(2010)</td>
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<td>Hill-Kujawa-</td>
<td>$\mathfrak{gl}_n(\mathbb{C})$</td>
<td>$M \otimes N \otimes V^\otimes d$</td>
<td>extend. degenerate Hecke alg</td>
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<td>Sussan (2009)</td>
<td>or $\mathfrak{sl}_n(\mathbb{C})$</td>
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<td>our case</td>
<td>$\mathfrak{gl}_{n</td>
<td>m}(\mathbb{C})$</td>
<td>$M \otimes N \otimes V^\otimes d$</td>
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The Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n|m)$ is the vector space $\text{Mat}_{n+m,n+m}(\mathbb{C})$ with the following $\mathbb{Z}_2$-grading

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A \in \text{Mat}_{n,n}, D \in \text{Mat}_{m,m} \right\}$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} | B \in \text{Mat}_{n,m}, C \in \text{Mat}_{m,n} \right\}$$

and the Lie brackets

$$[x, y] = xy - (-1)^{i \cdot j}yx, \quad \forall x \in \mathfrak{g}_i, y \in \mathfrak{g}_j$$
$V = \mathbb{C}^{n+m}$: column vectors of height $n + m$

$\mathfrak{g}$ acts by matrix multiplication.

Polynomial representations: irreducible summands of $V^\otimes d$

(Sergeev, Berele-Regev) are indexed by Young diagrams inside

a $(n|m)$-hook.
For $gl(1|3)$:
Let the degenerate two-boundary braid group $\mathcal{G}_d$ be generated by

$$\mathbb{C}[x_1, \ldots, x_d], \quad \mathbb{C}[y_1, \ldots, y_d], \quad \mathbb{C}[z_0, \ldots, z_d], \quad \mathbb{C}\Sigma_d$$

under further relations.

**Theorem**

Let $M, N$ be objects in category $\mathcal{O}$ of $\mathfrak{gl}(n|m)$. There is a well-defined action

$$\mathcal{G}_d \rightarrow \text{End}_{\mathfrak{gl}(n|m)}(M \otimes N \otimes V^\otimes d)$$
The (two boundary) extended degenerate Hecke algebra $\mathcal{H}_d^{ext}$ is a quotient of $\mathcal{G}_d$ under further relations.

**Theorem**

Let $L(\boxed{ })$ and $L(\boxed{\boxed{\quad}\quad})$ be two irreducible $\mathfrak{g}$-modules labeled by arbitrary rectangles inside the $(n, m)$-hook, the above action induces a further action

$$\rho : \mathcal{H}_d^{ext} \rightarrow \mathcal{H}_d = \text{End}_{\mathfrak{g}l(n|m)}(L(\boxed{\quad}) \otimes L(\boxed{\boxed{\quad}\quad}) \otimes V^{\otimes d})$$
\[ \mathfrak{gl}(n|m) \sim L(\begin{array}{c} \text{hook tableau} \\ \text{according} \\ \text{to a combinatorial rule} \end{array}) \otimes L(\begin{array}{c} \text{hook tableau} \\ \text{according} \\ \text{to a combinatorial rule} \end{array}) \otimes V^\otimes d \oplus L(\lambda) \otimes \mathcal{L}^\lambda \]

where the isomorphism is as \((\mathfrak{gl}(n|m), \mathcal{H}_d)\)-bimodules.
Introduction

Background

Results

Two Boundary
Theorem

$\mathcal{L}^\lambda$ admits a basis

$$\{v_T \mid T: \text{semistandard tableaux of the skew shape } \lambda/\mu\}$$

where $\mu$ is a diagram inside $\lambda$ based on certain combinatorial rules.

Furthermore, the polynomial generators $z_i$ act by eigenvalues

$$z_o.v_T = \alpha + \beta|\mathcal{B}| + \sum_{b \in \mathcal{B}} 2c(b)$$

$$z_i.v_T = c(i)$$

Where $c(*)$ denotes the content of the box, $\mathcal{B}$ is a certain set of boxes in $\lambda$. 
\[ \rho : \mathcal{H}_d^{ext} \rightarrow \mathcal{H}_d = \text{End}_{\mathfrak{gl}(n|m)}(L(\begin{array}{c} \square \\ \square \end{array})) \otimes L(\begin{array}{c} \square \\ \square \end{array}) \otimes V^{\otimes d} \]

**Theorem**

\[ \text{Res}_{\mathcal{H}_d^{ext}}^{\mathcal{H}_d} \mathcal{L}^\lambda \text{ is irreducible. Therefore, } \rho(\mathcal{H}_d^{ext}) \text{ is a large subalgebra of } \mathcal{H}_d. \]
An explicit example..