Braid group actions on $D^{\mathrm{b}}(\mathcal{O})$

Fabian Lenzen

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- principal block $\mathcal{O}_0$ of BGG category for $\mathfrak{sl}_n$.
- Vermas $M(w \cdot 0)$, simples $L(w \cdot 0)$, projectives $P(w \cdot 0)$.

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Theorem (Rouquier)

Br_n acts on $D^{\mathrm{b}}(\mathcal{O}_0)$ via $\mathbf{L}\,\mathrm{Sh}_s$.

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Theorem (Seidel, Thomas)

Br_n acts on $D^{\mathrm{b}}(\mathcal{O}_0)$ via T_{E_i}.

Question

Are there E_i's such that $T_{E_i} \cong \mathbf{L}\,\mathrm{Sh}_{s_i}$?

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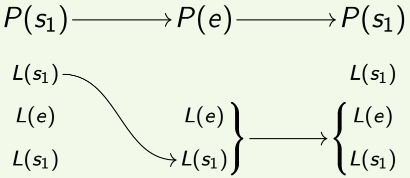
Example (spherical objects in $D^{\mathrm{b}}(\mathcal{O}_0)$ for $\mathfrak{sl}_2$)

$P(s_1)$ has $\mathbf{C}[\varepsilon]/(\varepsilon^2)$ as endomorphisms:



Hence $\{P(s_1)\}$ is an A_1-collection for $d = 0$.

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- $\{P(s)\}$ *is* 0*-spherical and* $T_{P(s)} \cong \mathbf{L}\,\mathrm{Sh}_s[-1]$
- $\{L(e)\}$ *is* 2*-spherical and* $T_{L(e)} \cong \mathbf{L}\,\mathrm{Sh}_s$

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Start with $\mathcal{O}_0$ for $\mathfrak{sl}_2$:

Caveat

- Remain not spherical under $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_n$:

$$P(s) \longrightarrow P(t) \longrightarrow P(s)$$

$$\begin{array}{c} L(s) \\ L(e) \quad L(st) \quad L(ts) \\ L(s) \quad L(t) \quad L(w_0) \\ L(st) \quad L(ts) \\ L(w_0) \end{array} \quad \begin{array}{c} L(t) \\ L(e) \quad L(st) \quad L(ts) \\ L(s) \quad L(t) \quad L(w_0) \\ L(st) \quad L(ts) \\ L(w_0) \end{array} \quad \begin{array}{c} L(s) \\ L(e) \quad L(st) \quad L(ts) \\ L(s) \quad L(t) \quad L(w_0) \\ L(st) \quad L(ts) \\ L(w_0) \end{array}$$

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- $D^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_2)) \subsetneq \nexists \subsetneq D^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_3))$ in which $P(s)$ is spherical.

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Theorem

For maximal parabolic subalgebra $\mathfrak{p} = \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix} \subset \mathfrak{sl}_n$,

- $\{P^{\mathfrak{p}}(s_1), \ldots, P^{\mathfrak{p}}(s_1 \cdots s_{n-1})\}$ *is an* A_{n-2}*-collection.*
- $T_{P^{\mathfrak{p}}(s_1 \cdots s_i)} \cong \mathbf{L}\,\mathrm{Sh}_{s_i}[-1]$.

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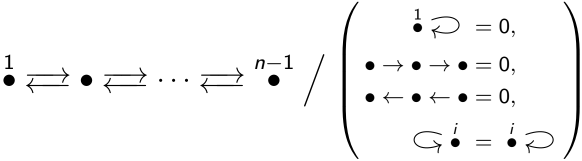

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Condition is necessary:
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Example: $\mathfrak{p} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$