What is a motivic gamma function?

1. Joint project with Spencer Bloch
   (bigger framework from Vasily Golyshev)

\[ L \text{ diff operator on } P_1 \setminus \{ \infty \} = U \]

finite set of singular points

\[ \Gamma_{0, \odot} (s) = \int_{0}^{\infty} \frac{\gamma(x)}{x} \, dx \quad \text{defined up to} \quad x \in \mathbb{C}, \text{ms}, m \in \mathbb{Z} \]

\[ \odot \text{ oriented closed path in } U \]
\[ \text{avoiding } 0 \text{ and } \infty \]

and contractible in \( P_1 \setminus \{ 0, \infty \} \)

\[ \gamma \text{ a solution to } L \gamma = 0 \]
\[ \text{defined in a neighbourhood of } \odot \]
and having trivial monodromy around \( \odot \)

Remarks:
1) \( \odot \sim \sum n_i [\odot_i] \), \( \gamma_i \sim \text{const} \cdot \gamma_i \)
gamma functions form a module over \( \mathbb{C}[e^{2\pi i s}] \)

2) \( \Gamma_{0, \odot} (s) \) is called motivic when
   \( L \) is of Picard-Fuchs type
   and \( \gamma(x) \) is a period function

\[ X \quad \gamma(x) = \int_{0}^{x} \omega(x) \]
\[ \gamma \text{ de Rham form} \]
\[ U / Q \text{ Betti cycle} \]

module of finite rank over \( \mathbb{Q}[e^{2\pi i s}] \)
Example \[ \Delta = (1 - x) \frac{d^2}{dx^2} - \frac{1}{2}, \quad \varphi(x) = (1 - x)^{-\frac{1}{2}} \sqrt{1 - x} \]

\[
\Gamma_{\delta}(s) = \int_{\delta} \frac{x^s}{\sqrt{1 - x}} \frac{dx}{x} = \int_{0}^{1} - e^{-2\pi i s} \int_{0}^{1} e^{-2\pi i s} \, dx = 2(1 - e^{-2\pi i s}) \frac{\Gamma(s) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + s)} \text{ entire, motivic}
\]

Applications:
- interpolation of recurrences
- Apéry constants (this talk)
- ...

Motivation:

\[ \gamma \rightarrow s_2 \]

\[ \delta \] path between two singular pts

\[ \{ \varphi_i^{(k)} \} \quad k = 1, 2 \]

basis in the space of solutions to \( \Delta \) near \( \delta \)

\[ [\delta] \varphi_i^{(k)} = \sum_j \varphi_j^{(2)} C_{ji} \]

connection matrix

Special choice: Frobenius basis

\[ C_0 \]

\[ \lambda \begin{pmatrix} 1 & 1 & 0 \\ 0 & \lambda & 0 \end{pmatrix} \]

Jordan block of \( [\delta_0] \)

local monodromy

\[ \lambda = e^{2\pi i p} \]

unique under the condition \( \varphi_i^{an}(0) \neq 0 \), \( i > 0 \)
Frobenius basis spans de Rham structure of the limiting MHS in mirror symmetry
quantum diff. e.g. of a Fano manifold \( \rightarrow \) Picard-Fuchs diff. e.g. \( L \) connection matrices contain info about the original Fano

Special case
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]
SS point + pseudo-reflection

Frobenius basis near \( s_1 \)
Apéry constants are \( x_0 = 1, x_1, \ldots, x_{n-1} \in \mathbb{C} \)
so \( [\ell] (F_{\ell}^{(n)} - x_0 F_0^{(n)}) \) is \( [s_2] \)-invariant

ambiguity:
\( s_i = 0 \)
\( F_{\ell} (x) = F_0 (x) \log (x) + F_1^{(\ell)} (x) \)
\( x_1 \sim x_1 + 2 \pi i \Rightarrow x_1 \in \mathbb{C} / 2 \pi i \mathbb{Z} \)

\( [s_1]^{m} F_0, \ldots, [s_1]^{m} F_{n-1} \) is also a Frobenius basis

\( [s_1] \sum F_0 (x) x^i (e^x) = e^{2 \pi i \varepsilon} \sum F_0 (e^x) x^i \)
\( \Rightarrow x^i (e) = \sum x^i e^i \) is defined up to \( * e \)

(note similarity with the ambiguity of \( \prod (s) \))

(In fact, one can define "higher" \( x_n, x_{n+1}, \ldots \) and "complete" \( x (q) \) to \( (almost) \) a gamma function; I am coming to this point after a short example.)
\[ L = D^3 - x (2D+1)(17D^2 + 17D + 5) + x^2 (D+1)^3 \]

\[ = x^3 (1 - 34x + x^2) \frac{d^3}{dx^3} + \ldots \]

\[ S = \{ 0, \infty, 17 \pm \sqrt{17^2 - 1} \} \]

\( x = 0 \) is MUM

\[ \Phi_0(x) = 1 + 5x + 73x^2 + \ldots \]

\[ \Phi_1(x) = \Phi_0(x) \log x + (12x + 210x^2 + \ldots ) \overline{\Phi_1^{\text{an}}(x)} \]

\[ \Phi_2(x) = \Phi_0(x) \frac{\log^2 x}{2!} + \Phi_1^{\text{an}}(x) \log x + (72x^2 + 2160x^3 + \ldots ) \overline{\Phi_2^{\text{an}}(x)} \]

\[ c = 17 - \sqrt{17^2 - 1} = 0.0234 \ldots \]

reflection point \[ C \quad (-1 \quad 0 \quad 0) \quad (-\infty \quad 0 \quad 1) \]

\( x_0 = 1 \quad x_1 = 0 \quad x_2 = -\frac{\pi^2}{3} \)

Lemma \[ L = \sum_{i=0}^{N} x^i \Phi_i(D) \]

polynomial diff operator

of order \( N \)

has MUM at \( x = 0 \) iff \( \Phi_0(D) = D^N \)

\( \Rightarrow \) all \( D^N \) have MUM at \( x = 0 \)

can construct higher Frobenius functions

( Golyshin, Zagier )

\[ \Phi_1(x), \Phi_2(x), \ldots \]

\[ \Phi(x, \varepsilon) = \sum_{i=0}^{\infty} \Phi_i(x) \varepsilon^i \]

\[ \Delta \Phi = \varepsilon^N x^2 \]

More interestingly: \( C = 0.0234 \ldots \)

seems to offer a reflection point for all \( D^N \)

G-Z compute

\( x_3 = \frac{17}{6} s(3) \), \( x_4 \), \ldots , \( x_{10} \), \( x_{11} \) involves \( s(3, 5, 3) \)!
Problem: understanding higher \( \zeta \)'s as periods?

geometric origin of \( D^k \) ?

\( k = 1 \) Mahler measure, normal functions
\( k = 2 \) ... more \( K \)-theory

**Theorem**

\[ c = 0.0294 \ldots \text{ remains semisimple for all } D^k, \quad k = 0, 1, 2, \ldots \]

\[ \Rightarrow \{ \sigma \} \text{ is a reflection} \]

Therefore all higher Apéry constants exist and in fact

\[ x(\varepsilon) = \sum_{i=0}^{\infty} x_i \varepsilon^i = \left( \frac{\varepsilon^{2\pi i}}{e^{2\pi i} - 1} \right)^3 \prod_{\sigma, \gamma} \psi(\varepsilon) \]

\[ \sigma = \sigma_0^{-3} \sigma_c (\sigma_0 \sigma_c)^3 \]

\[ \psi = \text{unique } [\sigma]-\text{invariant solution to } L \text{ normalized so that } \prod_{\sigma, \gamma} \psi(0) = 1 \]

\[ = -\frac{1}{3} \cdot \frac{1}{2\pi i} \Phi_0 - \frac{1}{2} \cdot \frac{1}{(2\pi i)^2} \Phi_1 + \frac{1}{3} \cdot \frac{1}{(2\pi i)^3} \Phi_2 \]

Remark: the presentation (RTS here) is non-canonical.

we can have \( P(e^{2\pi i}) x(x) = e^N \prod_{\sigma, \gamma} \psi(\varepsilon) \)

a poly with root 1 of multiplicity \( 2N \)

the canonical presentation is

\[ x(x) = e^N \int_0^1 x^2 \delta(x) \frac{dx}{x} \]

\( \delta = (\text{uniquely normalized}) \]

\[ [\sigma_c] \delta = -\delta \]

reflection!
What is our benefit? in understanding $\mathcal{X}_i$, is as periods

Theorem $\Rightarrow \mathcal{X}_i = \text{linear combinations of iterated integrals}$

(As Francis Brown and Richard Hein just explained to us, iterated integrals are periods of a relative completion!)

... work in progress

$$
\Gamma_{\sigma_i, y}^{(n)}(0) = \int_{\sigma_i} \log^k(x) \, y(x) \frac{dx}{x}
$$

**Lemma** $[\sigma] \quad \mathcal{D}^{-k} y_\sigma(x) = \mathcal{D}^k y_\sigma(x) + \sum_{j=0}^{k-1} \frac{(\gamma^{(j)}(0) \log^{k-1-j}(x))}{j! (k-1-j)!}$

indefinite iterated integral

**Example** polylogs $y = \frac{1}{1-x}$ $L_5 = (1-x) \frac{d}{dx} - 1$

$$
\Gamma_{\sigma_1, y}^{(5)}(1) = \int_{\sigma_1} \frac{x^5}{1-x} \frac{dx}{x} = -2\pi i
$$

What does this tell us?

$[\sigma_1] \quad \text{Li}_k(x) = \text{Li}_k(x) - 2\pi i \frac{\log^{k-1}(x)}{(k-1)!}$