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What is .. a motivic gamma function?

• joint project with Spencer Bloch
• started on ideas from Vasily Golyshev (bigger framework)

L diff operator on $\mathbb{P}^1 \setminus S = \mathcal{U}$

S ↑ finite set of singular points

⇓

entire functions

$$\Gamma_{\sigma,\varphi}(s) = \int_\sigma x^{+s} \varphi(x) \frac{dx}{x}$$

ambiguity: defined up to $\times e^{2\pi i m s}$ $m \in \mathbb{Z}$

σ oriented closed path in $\mathcal{U}$ avoiding 0 and ∞ and contractible in $\mathbb{P}^1 \setminus \{0, \infty\}$

possible generalization: σ with boundary in $S \cup \{0,\infty\}$

φ a solution to $L\varphi = 0$ defined in a neighbourhood of σ and having trivial monodromy around σ

Remarks: 1) $\sigma \rightsquigarrow \sum n_i [\sigma_i]$, $\varphi_i \rightsquigarrow \text{const}_i \cdot \varphi_i$
gamma functions form a module over $\mathbb{C}[e^{2\pi i s}]$

2) $\Gamma_{\sigma,\varphi}(s)$ is called motivic when
L is of Picard-Fuchs type
and $\varphi(x)$ is a period function

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathcal{U} / \mathbb{Q} \end{array} \qquad \varphi(x) = \int_{\gamma(x)} \omega(x)$$

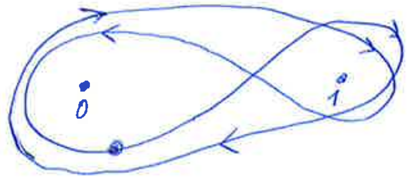
$\omega(x)$ ↳ de Rham form

$\gamma(x)$ ↳ Betti cycle

module of finite rank over $\mathbb{Q}[e^{2\pi i s}]$

Example $\quad L = (1-x)\dfrac{d}{dx} - \dfrac{1}{2} \qquad \varphi(x) = (1-x)^{-1/2}$

σ



$$\Gamma_{\sigma,\varphi}(s) = \int_\sigma \frac{x^{s}}{\sqrt{1-x}}\,\frac{dx}{x} =$$

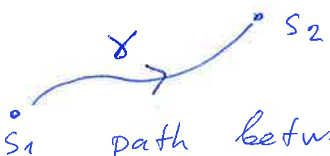
$$= \int_0^1 - \int_1^0 - e^{-2\pi i s}\int_0^1 + e^{-2\pi i s}\int_1^0 = 2\left(1-e^{-2\pi i s}\right)\int_0^1 \frac{x^s}{\sqrt{1-x}}\,\frac{dx}{x}$$

$$= 2\left(1-e^{-2\pi i s}\right)\frac{\Gamma(s)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+s)}$$

entire, motivic

Applications:
- interpolation of recurrences
- Apéry constants (this talk)
- ...

Motivation:



path between two singular pts

$\{\varphi_i^{(k)}\}$ $\quad k = 1, 2$

bases in the space of solutions to L near s_k

$$[\gamma]\,\varphi_i^{(1)} = \sum_j \varphi_j^{(2)}\, C_{ji}$$

connection matrix

special choice: Frobenius basis

regular singularity

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ 0 & & & \ddots \end{pmatrix}_{k\times k}$$

Jordan block of $[\sigma_0]$ local monodromy

$$\lambda = e^{2\pi i \rho}$$

$$\left(\varphi_0^{an}(x), \dots, \varphi_{k-1}^{an}(x)\right) x^{\rho + \underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{pmatrix}}_{\frac{1}{2\pi i}\log[\sigma_0]}}$$

$$\varphi_0(x) = x^\rho\,\varphi_0^{an}(x)$$

$$\varphi_1(x) = x^\rho\left(\varphi_0^{an}(x)\log(x) + \varphi_1^{an}(x)\right)$$

$$\varphi_2(x) = x^\rho\left(\varphi_0^{an}(x)\frac{\log^2(x)}{2!} + \varphi_1^{an}(x)\log x + \varphi_2^{an}(x)\right)$$

$$\dots$$

unique under the condition $\varphi_0^{an}(0) = 1$, $\varphi_i^{an}(0) = 0 \quad i > 0$

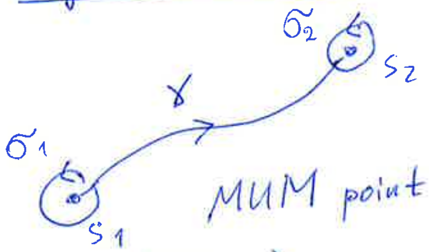
Frobenius basis spans de Rham structure of the limiting MHS

in mirror symmetry

quantum diff. eq. of a Fano manifold $\rightsquigarrow$ Picard-Fuchs diff. eq. L

connection matrices contain info about the original Fano

<u>Special case</u>



$$[\sigma_2] \sim \begin{pmatrix} * & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix}$$ SS point + pseudo-reflection

$$[\sigma_1] \sim \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & \ddots & \\ 0 & & \ddots & \ddots \end{pmatrix}$$

$\varphi_0, \dots, \varphi_{N-1}$ Frobenius basis near s_1

<u>Apéry constants</u> are $\varkappa_0 = 1, \varkappa_1, \dots, \varkappa_{N-1} \in \mathbb{C}$

s.t. $[\gamma]\left(\varphi_i(x) - \varkappa_i \varphi_0(x)\right)$ is $[\sigma_2]$-invariant

ambiguity: $s_1 = 0$ $\quad \varphi_1(x) = \varphi_0(x)\log(x) + \varphi_1^{an}(x)$

$\varkappa_1 \rightsquigarrow \varkappa_1 + 2\pi i\, m$, $m \in \mathbb{Z}$ $\qquad \varkappa_1 \in \mathbb{C}/2\pi i\mathbb{Z}$

$[\sigma_1^m]\varphi_0, \dots, [\sigma_1^m]\varphi_{N-1}$ is also a Frobenius basis

$$[\sigma_1] \sum \varphi_i(x)\,\varepsilon^i \overset{(\varepsilon^N)}{=} e^{2\pi i \varepsilon} \sum \varphi_i(\varepsilon)\,\varepsilon^i$$

$$\Rightarrow \varkappa(\varepsilon) = \sum \varkappa_i\, \varepsilon^i \text{ is defined up to } * e^{2\pi i m \varepsilon}$$

(note similarity with the ambiguity of $\Gamma(s)$!)

(In fact, one can define "higher" $\varkappa_N, \varkappa_{N+1}, \dots$ and "complete" $\varkappa(\varepsilon)$ to (almost) a gamma function! I am coming to this point after a short example.)

Example

$$L = D^3 - x(2D+1)(17D^2+17D+5) + x^2(D+1)^3$$

$$= x^3(1-34x+x^2)\frac{d^3}{dx^3} + \dots$$

$$S = \{0, \infty, 17 \pm \sqrt{17^2-1}\}$$

$x=0$ is MUM

$$\varphi_0(x) = 1 + 5x + 73x^2 + \dots$$

$$\varphi_1(x) = \varphi_0(x)\log x + (\underbrace{12x + 210x^2 + \dots}_{\varphi_1^{an}(x)})$$

$$\varphi_2(x) = \varphi_0(x)\frac{\log^2 x}{2!} + \varphi_1^{an}(x)\log x + (\underbrace{72x^2 + 2160x^3 + \dots}_{\varphi_2^{an}(x)})$$

0

$c = 17 - \sqrt{17^2-1} = 0{,}0294\dots$

reflection point $[\sigma_c] \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\varkappa_0 = 1 \qquad \varkappa_1 = 0 \qquad \varkappa_2 = -\frac{\pi^2}{3}$$

Lemma $L = \sum_{i=0}^{r} x^i P_i(D)$ polynomial diff operator of order N

has MUM at $x=0$ iff $P_0(D) = D^N$

$\Rightarrow$ all $D^k L$ have MUM at $x=0$

can construct higher Frobenius functions

(Golyshev - Zagier)

$\varphi_N(x), \varphi_{N+1}(x), \dots$ $\qquad \Phi(x,\varepsilon) = \sum_{i=0}^{\infty} \varphi_i(x)\varepsilon^i$

$$L\Phi = \varepsilon^N x^\varepsilon$$

More interestingly: $c = 0{,}0294\dots$ seems to stay a reflection point for all $D^k L$ ...

G-Z compute

$\varkappa_3 = \frac{17}{6}\zeta(3)$, $\underbrace{\varkappa_4, \dots, \varkappa_{10}}_{\text{rational combinations of ZVs of weight } 4, \dots, 10}$, $\varkappa_{11}$ involves $\zeta(3,5,3)$!

Problem: understanding higher $\varkappa_i$'s as periods?

geometric origin of $D^k L$?

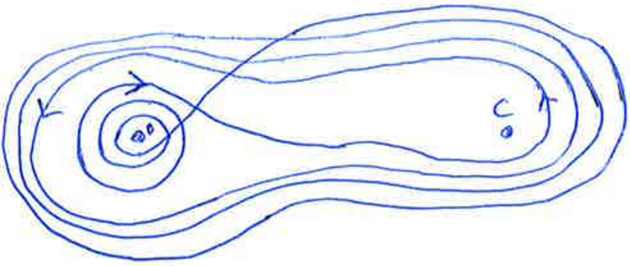
$k=1$ Mahler measure, normal functions

$k=2$ ... more K-theory

Theorem $c = 0.0294\ldots$ remains semisimple for all $D^k L$, $k = 0,1,2,\ldots$ ($\Rightarrow$ $[\sigma_c]$ is a reflection).

Therefore all higher Apéry constants exist and in fact

$$\varkappa(\varepsilon) = \sum_{i=0}^{\infty} \varkappa_i \varepsilon^i = \left(\frac{2\pi i\, \varepsilon}{e^{2\pi i \varepsilon} - 1} \right)^3 \Gamma_{\sigma,\psi}(\varepsilon)$$



$$\sigma = \sigma_0^{-3} \sigma_c (\sigma_0 \sigma_c)^3$$

ψ = unique $[\sigma]$-invariant solution to L normalized so that $\Gamma_{\sigma,\psi}(0) = 1$

$$= -\frac{1}{9} \cdot \frac{1}{2\pi i} \varphi_0 - \frac{1}{2} \cdot \frac{1}{(2\pi i)^2} \varphi_1 + \frac{1}{3} \cdot \frac{1}{(2\pi i)^3} \varphi_2$$

Remark: the presentation (RHS here) is non-canonical

we can have $P(e^{2\pi i \varepsilon}) \varkappa(\varepsilon) = \varepsilon^N \Gamma_{\sigma,\psi}(\varepsilon)$

a poly with root 1 of multiplicity $\geq N$

the canonical presentation is

$$\varkappa(\varepsilon) = \varepsilon^N \int_0^c x^\varepsilon \delta(x) \frac{dx}{x}$$

δ = (uniquely normalized) $[\sigma_c]\delta = -\delta$

reflection!

What is our benefit? in understanding $\mathcal{X}_i$'s as periods

Theorem $\Rightarrow$ $\mathcal{X}_i$ = linear combinations of iterated integrals

(As Francis Brown and Richard Hain just explained to us, iterated integrals are periods of a relative completion!)

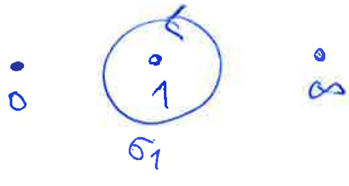
... work in progress

$$\Gamma_{\sigma,\varphi}^{(k)}(0) = \int_{\sigma} \log^{k}(x)\, \varphi(x)\, \frac{dx}{x}$$

<u>Lemma</u> $$[\sigma]\, D^{-k}\varphi(x) = D^{-k}\varphi(x) + \sum_{j=0}^{k-1} (-1)^j \frac{\Gamma_{\sigma,\varphi}^{(j)}(0)}{j!} \frac{\log^{k-1-j}(x)}{(k-1-j)!}$$

indefinite iterated integral

<u>Example</u> polylogs $\varphi = \frac{1}{1-x}$ $\qquad L = (1-x)\frac{d}{dx} - 1$



$$\Gamma_{\sigma_1,\varphi}(s) = \oint \frac{x^s}{1-x}\frac{dx}{x} = -2\pi i$$

What does this tell us?

$$[\sigma_1]\, Li_k(x) = Li_k(x) - 2\pi i \frac{\log^{k-1}(x)}{(k-1)!}$$