 [periods in number theory algebraic geometry and physics. wordpress.com] - calendar / blog

Andrey Levin Modular arrangements.

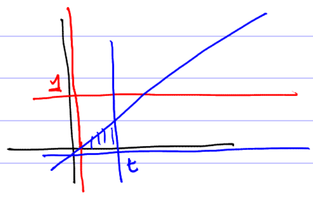
(joint work with Nina Sakharova).

Classical result: $Y = SL_2(\mathbb{Z})\backslash \mathfrak{H}$, $Y \times Y \supset T_n^\circ = \{(\tau_1,\tau_2) \mid \tau_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau_2,\ ad-bc=n\}$
a,b,c,d coprime.

Modular arrangement = finite collection of T_n°'s.

Qns: • combinatorics of intersections?
• π_1 of complement?
• Aomoto modular dilogs?

Ex, $\mathbb{C}^2$:



$\leftrightarrow \operatorname{dlog} x \wedge \operatorname{dlog}(1-y) = \Omega$

$$\pm \int_\Delta \Omega = Li_2(t) = \sum_{n \geq 1} \frac{t^n}{n^2}$$

– single-valued version $\mathbb{L}_2 = Li_2(t) - \overline{Li_2(t)} - \overset{?}{\left(\tfrac{1}{2}\right)} \log|t| \left(Li_1(t) - \overline{Li_1(t)} \right)$

$D_2 = \frac{1}{2i}\mathbb{L}_2$.

– geom. description of D_2: $\Xi = \operatorname{dlog} y \wedge \operatorname{dlog}(x-t) + \operatorname{dlog}(x-t) \wedge \operatorname{dlog}(x-y)$
$+ \operatorname{dlog}(x-y) \wedge \operatorname{dlog}(y)$

$$\int_{\mathbb{C}^2} \Omega \wedge \Xi \sim \pm 2(2\pi i)^2 \left(\mathbb{L}_2(t) + \log|t|\, \mathbb{L}_1(t) \right)$$
$\mathbb{L}_1 = \operatorname{Re} Li_1$

$$\shortparallel$$

$$\int_{\mathbb{C}^2} (\Omega - \bar{\Omega}) \wedge \Xi$$

$\downarrow$

get 5 boundary terms, which are 1-dim'l integrals, nonzero only on $\{x=y\}$ and $\{x=t\}$.

$\downarrow$

$\varphi(x, 1-x)$

$\downarrow$

$\mathbb{L}_2$.

Goncharov observed:

$$d\Big(\overbrace{\log|f|\,(\partial - \bar{\partial})\log|g| - }^{\varphi(f,g)} \quad f \leftrightarrow g \Big)$$

$= (*)\left(\partial\log f \wedge \partial \log g - \overline{\partial \log f \wedge \partial \log g} \right)$

i.e. $\Omega - \bar{\Omega}$ is exact

0^{th} example: $\displaystyle\int_{\mathbb{C}} \frac{dz}{z} \wedge \overline{\left(\operatorname{dlog}(t-z) - \operatorname{dlog}(1-z) \right)}$

$= \displaystyle\int_{\mathbb{C}} 2\operatorname{dlog}|z| \wedge \left(\text{—}'' \text{—} \right)$

$= \lim$ of $\int$ over $\left\{ \begin{matrix} |z|, |z-1|, |z-t| \geq r \\ |z| \leq R \end{matrix} \right\}$

$= \lim \displaystyle\int_{4 \text{ circles}} \log|z|^2 \cdot \overline{\operatorname{dlog}(t-z) - \operatorname{dlog}(1-z)}$

$= 2\pi i \log|t|^2$.

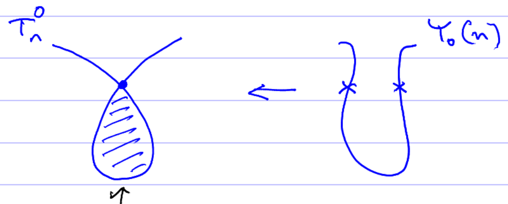
$\mathcal{O}^{bis}$: C curve, p_1, p_2 $\quad \omega \quad \mathrm{res}_{p_i}\omega = \pm 1$ (normalized st. $\int \omega \bar{\nu} = 0$ $\forall$ regular ν)

q_1, q_2 $\quad \psi \quad \mathrm{res}_{q_i}\psi = \pm 1$

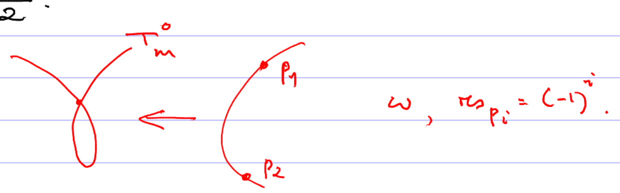
$$\int_C \omega \wedge \bar{\psi} = \langle p_1 - p_2, q_1 - q_2 \rangle_\infty .$$

Modular case: T_n^0 singular (if $n \geq 2$, $(\sqrt{1-n}, \sqrt{1-n}) \in T_n^{0,\mathrm{sing}}$ as $\begin{pmatrix} \pm 1 & 1-n \\ 1 & \pm 1 \end{pmatrix} \sqrt{1-n} = \sqrt{1-n}$)
($\forall n \geq 2$)

$n=2$: use $\frac{1+\sqrt{-7}}{2}$.



bounded by some path between 2 singular points.



2-form Ω with pole along T_m^0, residue $= \omega$.

$$\sum_{\text{"}SL_2 \times SL_2\text{"}} \frac{\bullet}{|\bullet|^{2s}}$$

Singularity of T_m: order $\mathcal{O} \ni \mu$ st. $N(\mu) = m$, μ primitive (not div by any integer > 1)

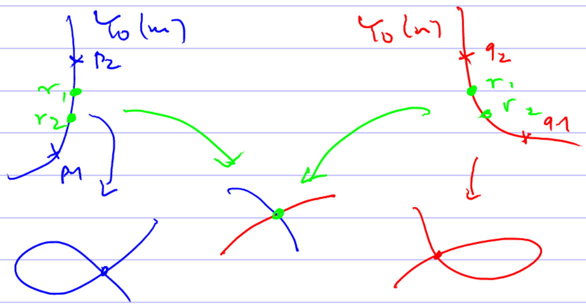
& of T_n: $\mathcal{O}' \ni \nu$ $\quad \rightsquigarrow$ 2-form $\Omega(\mu)$ on Y^2

Assume $X_0(m), X_0(n)$ have genus 0.

$$\int_{Y^2} \Omega(\mu) \wedge \overline{\Omega(\nu)}$$ should be some kind of dilogarithm. $D_2(?)$

where $? \in H(\mathcal{O}).H(\mathcal{O}')$ (or pre-Bloch gp)

Picture:-



as curves are rational,

$$\sum_{r' \neq r} CR_{Y_0(m)}(p_1 p_2 r, r') \wedge CR_{Y_0(n)}(q_1 q_2 r, r')$$

$\in \mathbb{C}^* \wedge \mathbb{C}^*$ is apparently

the image of the element $?$ under $[x] \mapsto x \wedge (1-x)$