Bowman-Bradley type identities for symmetrised MZV’s

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1. Background: finite MZV’s and symmetrised MZV’s

Finite MZV’s defined by Hoffman, Zhao, and others as follows

$$\zeta_p(k_1, \ldots, k_r) = \sum_{0 < m_1 < \cdots < m_r < p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \pmod{p},$$

truncating before p in the denominator.

Zagier then considered all $\zeta_p$ simultaneously, modulo ‘finite differences’, to define:

$$\zeta^A(k_1, \ldots, k_r) := (\zeta_p(k_1, \ldots, k_r) \pmod{p})_{p \in A} \equiv \prod_p \mathbb{Z}/p\mathbb{Z} / \bigoplus_p \mathbb{Z}/p\mathbb{Z}$$

Since $\mathbb{Q} \to A$ diagonally, this is a $\mathbb{Q}$-algebra. These MZV’s are defined for all $k_i \in \mathbb{Z}$, but get some mixing of weight

$$\zeta_p(-1, 3) = \sum_{0 < m < n < p} \frac{m}{n^3} = \sum_{0 < n < p} \frac{1}{n} \frac{n-1}{n^3} = \frac{1}{2} \zeta_p(1) - \frac{1}{2} \zeta_p(2)$$

Imposing $k_i \geq 1$ fixes this, and allows us to define the space of weight $k$ finite MZV’s, write $Z_{A,k}$. Experiments suggest

$$\dim_{\mathbb{Q}} Z_{A,k} = d_k - 3$$

Comparing dimensions, via $d_{k-3} = d_k - d_{k-2}$ suggests maybe

$$Z_{A,k} \cong \mathbb{Z} / \pi^2 \mathbb{Z}$$

A suggestion for defining this isomorphism is via the symmetrised MZV’s

$$\zeta^S = \sum_{i=0}^{r} (-1)^{k_{i+1} + \cdots + k_r} \zeta^*(k_1, \ldots, k_i) \zeta^*(k_r, \ldots, k_{i+1}),$$

for $\zeta^* = \omega, \ast$-regularisation.

**Proposition 1.1** (Kaneko-Zagier).

$$\zeta^S - \zeta^S \ast \in \pi^2 \mathbb{Z},$$

so $\zeta^S = \zeta^S \ast \pmod{\pi^2}$ is well defined.

Then conjectural isomorphism $Z_A \to \mathbb{Z} / \pi^2 \mathbb{Z}$ is given via

$$\zeta_S(k_1, \ldots, k_r) \mapsto \zeta_A(k_1, \ldots, k_r)$$

On the finite side, we have

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Theorem 1.2 (Bowman-Bradley type - [SW16]). Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be odd integers, and \( c_1, \ldots, c_m \) be even integers, all \( \geq 1 \). Then
\[
\sum_{(\sigma, \tau) \in S_n^2} \zeta^A(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}) \tilde{\tilde{\mu}}\{c_1\} \tilde{\tilde{\mu}} \cdots \tilde{\tilde{\mu}}\{c_m\}
= \sum_{(\sigma, \tau) \in S_n^2} \sum_{\rho \in S_m} \zeta^A(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}) \tilde{\tilde{\mu}}\{c_{\rho(1)}, \ldots, c_{\rho(m)}\}
= 0
\]

Remark 1.3. Here \( \tilde{\tilde{\mu}} \) means shuffle of the MZV arguments (not the iterated integrals), i.e.
\[
\{a_1, a_2, \ldots, a_p\} \tilde{\tilde{\mu}}\{b_1, b_2, \ldots, b_q\} = a_1\{\{a_2, \ldots, a_p\} \tilde{\tilde{\mu}}\{b_1, b_2, \ldots, b_q\}\} + b_1\{\{a_1, a_2, \ldots, a_p\} \mu\{b_2, \ldots, b_q\}\}.
\]
For example,
\[
\zeta(\{2, 2\} \mu \{3, 5\}) = \zeta(2, 2, 3, 5) + \zeta(2, 3, 2, 5) + \zeta(2, 3, 5, 2) + \zeta(3, 2, 2, 5) + \zeta(3, 2, 5, 2) + \zeta(3, 5, 2, 2)
\]
Goal: corresponding result for symMZV’s.

Remark 1.4. Some results already by Muneta (Kyushu MZV seminar)
\[
\zeta^S(\{1, 3\}^n \tilde{\mu}\{2\}^m) = \binom{m + n}{n} (-1)^{n2^{2m+2n+1}} \frac{2^{2m+4n}}{(2m + 4n + 1)!} \equiv 0 \pmod{\pi^2}
\]
(Here \( \mu, *\)-regularisation are equal because there is no consecutive 1, 1 in the result.)
Murahara also has some unwritten results.

2. A general Bowman-Bradley ‘type’ identity

Theorem 2.1. Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be odd integers, and \( c_1, \ldots, c_m \) be even integers, all \( \geq 1 \). Then
\[
\sum_{(\sigma, \tau) \in S_n^2} \zeta^S(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}) \tilde{\tilde{\mu}}\{c_1\} \tilde{\tilde{\mu}} \cdots \tilde{\tilde{\mu}}\{c_m\}
= \sum_{(\sigma, \tau) \in S_n^2} \sum_{\rho \in S_m} \zeta^S(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}) \tilde{\tilde{\mu}}\{c_{\rho(1)}, \ldots, c_{\rho(m)}\}
= \sum_{(\sigma, \tau) \in S_n^2} \sum_{B_1, \ldots, B_k} \sum_{B \in \Pi_{\leq 2}(m)} (-1)^n2^\#B \zeta(\{a_{\sigma(1)} + b_{\tau(1)}, \ldots, a_{\sigma(n)} + b_{\tau(n)}\}) \tilde{\tilde{\mu}}\{c_{B_1}\} \tilde{\tilde{\mu}} \cdots \tilde{\tilde{\mu}}\{c_{B_k}\}.
\]
Here we employ the following notation
\[
\Pi_{\leq 2}(m) := \{ \text{all partitions } (B_1, \ldots, B_k) \text{ of } \{ 1, \ldots, m \} \text{ with } \#B_i \leq 2 \}, \text{ and }
\]
\[
c_{B_j} := \sum_{k \in B_j} c_k.
\]

Corollary 2.2. The result vanishes modulo \( \pi^2 \), which matches the expectation under the \( \zeta^A \leftrightarrow \zeta^S \) correspondence.

Proof. For arbitrary odd \( a_i, b_i \), even \( c_i \), we see the result is \( 0 \pmod{\pi^2} \): After summing over \( (\sigma, \tau) \in S_n \), the \( \zeta \) is symmetric in all arguments. Hence by the symmetric sum formula, we can write the result as a polynomial in
\[
\zeta(\{a_{\sigma,\tau,1,j}(a_{\sigma(1)} + b_{\tau(j)}) + \beta_k c_{B_k}\})
\]
Since \( a_i + b_j \) and \( \sum c_i \) are all even, the result is an even zeta, which vanishes modulo \( \pi^2 \). \( \square \)
Sketch of Theorem. Purely combinatorial, and by induction. Very similar to the finite case. Case $m = 0$ corresponds to the following stuffle-algebra identity

$$
\sum_{(\sigma, \tau) \in S_n^2} \left\{ \sum_{i=0}^{n} z_{a_{\sigma(i)}} \cdots z_{a_{\sigma(i+1)}} z_{b_{\tau(1)}} \cdots z_{b_{\tau(n)}} \right\} = \sum_{(\sigma, \tau) \in S_n^2} (-1)^n z_{a_{\sigma(1)} + b_{\tau(1)}} \cdots z_{a_{\sigma(n)} + b_{\tau(n)}}
$$

which is also proven by induction.

Then use the stuffle-product result

$$
\zeta^S(k)\zeta^S(l) = \zeta^S(k \ast l)
$$

and relate this to $\tilde{\mu}$ as follows

$$
\zeta^S(3\tilde{\mu}\{c\}) = \zeta^S(k \ast c) - \sum_i \zeta(k_1, \ldots, k_i + c, \ldots, k_n).
$$

This allows us to shuffle in a single $c$ at a time, to obtain the result. One obtains two level $m$ versions with variables either $a_i/b_i + c_{m+1}$, or $a_i/b_i, c_j + c_{m+1}$.

3. Corollaries and evaluations

From this can set $a_i = a$, $b_i = b$, to obtain

**Corollary 3.1** (Bowman-Bradley).

$$
\zeta^S([a, b]^n\tilde{\mu}\{c\})^m = \sum_{i=0}^{[m/2]} (-1)^n 2^{m-2i} \zeta([a + b]^n\tilde{\mu}\{c\}^m - 2i\tilde{\mu}\{2c\})^i = 0 \pmod{\pi^2}.
$$

When $m = 0$, we obtain

$$
\zeta^S([a, b]^n) = (-1)^n \zeta([a + b]^n),
$$

which can be explicitly evaluated in each case using generating series results about $\zeta(\{even\})$.

To go to higher $m$, we need to evaluate combinations like $\zeta([p]^k\tilde{\mu}\{q\}^l\tilde{\mu}\{r\}^m)$. I’m not aware of any such results so far, but I can conjecture the following

**Observation 3.2.** For any $a, b, c \in \mathbb{Z}_{\geq 0}$, the following evaluation appears to hold

$$
\zeta(\{2^a\tilde{\mu}\{4^b\tilde{\mu}\{6^c\}} = \frac{2^{1+2b+6c}(b + 2c)!((1 + a + 2b + 4c)!}{(2 + 2a + 4b + 6c)!}(b + 2c)\left(\frac{1 + a + 2b + 4c}{a}\right)
$$

**Corollary 3.3.** (Assuming the above is accurate), the following evaluations hold

(Muneta)

$$
\zeta^S([1, 3]^n\tilde{\mu}\{2\})^m = \binom{m + n}{n} \left(\frac{-1)^n 2^{m+2n+1}}{(2m + 4n + 2)!}\right)^{2^n} 2^{mn+4n}
$$

$$
\zeta^S([3, 3]^n\tilde{\mu}\{2\})^m = \frac{1}{2n + 1} \binom{2n + m}{m} \left(\frac{-1)^n 2^{m+6n+1}}{(2m + 6n + 2)!}\right)^{2^n} 2^{mn+6n}
$$

**Proof.** The resulting binomial sums can be evaluated using the WZ-method. \hfill \Box

**Remark 3.4.** Not sure if there is a nice generating series proof of the above observation; the naïve generating series obtained by generalising the $\zeta([a]^n)$ evaluation gives $\zeta([a]^n)\zeta([b]^l)$ type results instead.
Using the symmetric sum theorem, I can recursively reduce a proof of the above to proving the following Bernoulli identities, neither of which seems particularly easy to prove.

\[ \sum_{n=0}^{a} \sum_{m=0}^{b} (-1)^{n} 2^{a+2nb} B_{4+2n+4na} \left( \frac{6 + 2a + 4b}{4 + 2n + 4nb} \right) \left( \frac{1 + a + 2b - n_a - 2nb}{a - n_a} \right) \left( \frac{n_a + nb}{n_a} \right) \]

\[ \frac{- (b + 1)}{2} \left( 3 + a + 2b \right) \]

\[ \sum_{n=0}^{a} \sum_{m=0}^{b} \sum_{c=0}^{e} (-1)^{n} 2^{a+2nb} B_{6+2n+4nb+6nc} \left( \frac{8 + 2a + 4b + 6c}{6 + 2n + 4nb + 6nc} \right) \left( \frac{1 + a + 2b + 4c - n_a - 2nb - 4nc}{a - n_a} \right) \left( \frac{b + 2c - nb - 2nc}{2c - 2n_c} \right) \left( \frac{n_a + nb + nc}{n_a, n_b, n_c} \right) \]

\[ \frac{2 + 2c}{3 + 2c} \left( 2 + b + 2c \right) \left( 5 + a + 2b + 4c \right) \]

(Murahara recently suggested a different recursion, and reduces this to a certain binomial sum identity. Hopefully this is more accessible.)

Beyond \( \zeta(2\tilde{\varpi}4\tilde{\varpi}6) \), one necessarily encounters \( \zeta(8) \), and like the evaluation of \( \zeta(\{8\}^n) \), these evaluations become more difficult to find and write.

**Theorem 3.5.** For \( n \geq 0 \), \( R_{\pm} = 64(17 \pm 12\sqrt{2}) = 4^3(1 \pm \sqrt{2})^4 \), and \( \sigma: \sqrt{2} \mapsto -\sqrt{2} \) the Galois automorphism of \( \mathbb{Q}(\sqrt{2}) \), we have

\[ \zeta(\{8\}^n \tilde{\varpi}(\{2\}^0)} = \frac{n^{\pm 1}}{(\sqrt{2} + 1)} \left\{ R_{\pm} \left( 12 + 8\sqrt{2} \right) \right\}^{\sigma \sim \text{Galois symmetrisation}} \]

\[ \zeta(\{8\}^n \tilde{\varpi}(\{2\}^1)} = \frac{n^{\pm 1}}{(\sqrt{2} + 1)} \left\{ R_{\pm} \left( 60 + 42\sqrt{2} + n(80 + 56\sqrt{2}) \right) \right\}^{\sigma} \]

\[ \zeta(\{8\}^n \tilde{\varpi}(\{2\}^2)} = \frac{n^{\pm 1}}{(\sqrt{2} + 1)} \left\{ R_{\pm} \left( 168 + 118\sqrt{2} + n(440 + 310\sqrt{2}) + n^2(272 + 192\sqrt{2}) \right) \right\}^{\sigma} \]

*Proof.* Proven using \( \zeta(\{8\}^n) \) as the base case, and summing up the Bernoulli sums using the generating series of Bernoulli polynomials. \( \square \)

**Observation 3.6.** One finds that \( \zeta(\{8\}^n \tilde{\varpi}(\{2\}^m)} \), \( m \) fixed, appears to satisfy a linear recurrence relation of order \( 2m + 2 \), whose characteristic equation factors as

\[ (\lambda - R_+)^m + (\lambda - R_-)^m + 1 = 0. \]

So by finding the first \( 2m + 2 \) instances, one obtains further candidate results like

\( \zeta(\{8\}^n \tilde{\varpi}(\{2\}^3)} \) = \( \frac{n^{\pm 1}}{(\sqrt{2} + 1)} \left\{ R_{\pm} \left( 360 + \frac{1015}{4} \sqrt{2} + n(\frac{3994}{3} + \frac{2819}{3} \sqrt{2}) + n^2(1608 + 1136\sqrt{2}) + n^3(\frac{1856}{3} + \frac{1312}{3} \sqrt{2}) \right) \right\}^{\sigma} \),

and a general form

\[ \zeta(\{8\}^n \tilde{\varpi}(\{2\}^m)} = \frac{n^{\pm 1}}{(\sqrt{2} + 1)} \left\{ R_{\pm} \sum_{j=0}^{m} \alpha_j n^j \right\}^{\sigma}, \]

some \( \alpha_j \in \mathbb{Q}(\sqrt{2}) \).

Unfortunately, not clear what the pattern is coefficients is. Moreover, some coefficients have large prime factors dividing their norm:

\[ N_{\mathbb{Q}(\sqrt{2})}\left( \frac{3994}{3} + \frac{2819}{3} \sqrt{2} \right) = 2^1 \cdot 3^{-2} \cdot 17 \cdot 1721. \]
3.1. Miscellaneous results

It doesn’t yet appear as if any analogue of cyclic insertion holds in general for symMZV’s. Numerically, I have checked how Bowman-Bradley for \( \zeta(\{1, 3\} \cup \{2\}) \) decomposes into \( \pi^* \)-pieces, but it doesn’t seem so well structured yet.

However

**Observation 3.7.** The following evaluation

\[
\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \zeta^S(\{2\}^{a_{\sigma(1)}}, 3, \{2\}^{a_{\sigma(2)}}, 3, \ldots, 3, \{2\}^{a_{\sigma(2n+1)}}) = 2^w t + 1 \frac{\pi^w}{(w t + 2)!}
\]

appears to hold.

So there might be something interesting here. . .

The proof of ‘generalised Bowman-Bradley’ for \( \zeta^S \) should give directly a different generalisation when \( c_i \) are arbitrary, and \( n = 0 \). Namely

\[
\sum \zeta^S(\{c_1\} \hat{\mu} \cdots \hat{\mu}\{c_m\}) = \sum_{B = B_1 \ldots B_k} \prod_i (1 + \hat{f}_{B_i}) \cdot \zeta(\{c_{B_1}\} \hat{\mu} \cdots \hat{\mu}\{c_{B_k}\}),
\]

where \( c_{B_i} = \sum n_i c_i \). Note, in particular, that the if any \( c_{B_i} \) is odd, the term vanishes. So one could write this as a sum over all partitions \( B \in \Pi_{\leq 2}(m) \) such that every \( c_{B_i} \equiv 0 \pmod{2} \).

Moreover, one can probably give a common generalisation (naturally with a more complicated expression), of these two results, to arbitrary \( a, b, c \).

Nevertheless, one can give results like

\[
\zeta^S(\{1\}^{\{2\}} \hat{\mu} \{3\}^{\{2n\}})
\]

\[
= 2^{n+1}(2^{n+1} - 1)! 6^n (\zeta(\{2\} \hat{\mu} \{6\}^n) + 4\zeta(\{4, 4\} \hat{\mu} \{6\}^{n-1}))
\]

\[
= 2^{n+7} 2^n (2 + n) \frac{n!(-1 + 2n)!}{(4 + 6n)!} \pi^{2 + 6n}
\]

\[
\equiv 0 \pmod{\pi^2}
\]

\[
\zeta^S(\{3\}^{\{2\}} \hat{\mu} \{5\}^{\{2n\}})
\]

\[
= 2^{n+1}(2^{n+1} - 1)! 6^n (\zeta(\{6\} \hat{\mu} \{10\}^n) + 4\zeta(\{8, 8\} \hat{\mu} \{10\}^{n-1})) \equiv 0 \pmod{\pi^2}
\]

References