Motivic MZV's and the cyclic insertion conjecture

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HIM Periods Trimester, Bonn

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Abstract: I will start by recalling two conjectural families of MZV identities proposed by Borwein-Bradley-Broadhurst-Lisonek, and by Hoffman. I will show how both of these conjectures can be unified into a larger conjectural family of identities by using the so-called block decomposition of iterated integrals introduced here.

Using the motivic MZV framework of Brown I will show that a symmetrised version of this conjecture holds up to $\mathbb{Q}$. This will give a proof of Hoffman's identity, up to $\mathbb{Q}$ and an improvement of the Bowman-Bradley theorem giving some progress towards the BBBL conjecture.

Outline

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1 Definitions and conjectures

2 Motivic MZV's and algebraic tools

3 Alternating block decomposition

4 Extra material (time permitting)

Definitions and conjectures

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Mutiple zeta values

Definition (MZV)

Multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

- Where $s_i \geq 1 \in \mathbb{Z}$
- For convergence $s_k \geq 2$

Also define

- Weight is sum $s_1 + \cdots + s_k$ of arguments
- Depth is number k of arguments

My convention on MZV's means that $\zeta(1,2) = \zeta(3)$. This fits better with the motivic MZV framework used later.

MZV relations

MZV satisfy *lots* of relations

- Duality relations
- Associator relations
- Derivation relations
- (Extended) Double shuffle relations
- . . .

Not always clear how to prove *explicit* relations from these.

Theme: progress towards and generalisation of some *explicit* conjectural families of identities

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Theme: progress towards and generalisation of some *explicit* conjectural families of identities

1. It is well known that MZV's satisfy a huge number of relations. At weight k there are (a priori) 2^{k-2} MZV's. But the number of $\mathbb{Q}$-linearly independent ones is much less. For example there are 64 weight 8 MZV arguments, but 4 linearly independent ones $\zeta(8), \zeta(2,3,3), \zeta(3,2,3), \zeta(3,3,2)$ for example.
2. The extended double shuffle relations come form comparing two different ways of multiplying MZV's (as series, or as integrals – later). It is expected that all linear MZV relations follow from double shuffle. It is known that the associator relations (coming from the Drinfel'd associator, see other talks) imply the double shuffle relations. But the converse is not known.
3. If I give you a specific MZV identity, it can be computationally hard to prove it from these known relations. (Can brute for it with some linear algebra, in particular cases.) But for an infinite family, the brute force approach will likely stop working, meaning proofs can be very difficult.

Zagier-Broadhurst Identity

Theorem (Zagier-Broadhurst, BBBL 2001)

For $n \geq 0 \in \mathbb{Z}$, *have*

$$\zeta(\{1,3\}^n) = \frac{1}{2n+1} \frac{\pi^{4n}}{(4n+1)!}$$

Proof (Sketch).

- Generalise to single variable *multiple polylogarithms*.
- Generating series satisfies a differential equation.
- Explicit solution in terms of ${}_2F_1$. Compare coefficients.

Combinatorial proofs have also been given. □

Zagier-Broadhurst Identity
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Explicit solution in terms of ${}_2F_1$. Compare coefficients.
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1. This identity was spotted by Zagier after some numerical computations of MZV (which in particular lead to the famous conjecture on the dimension d_k of the weight k MZV's, namely $d_k = d_{k-2} + d_{k-3}$, $d_0 = 1, d_1 = 0$.
2. The first proof of this identity was given by Broadhurst, by lifting the putative identity to certain single variable multiple polylogarithms, and assembling the results in the a generating series.
3. One can check that this generating series satisfies a certain differential equation. One can give an explicit solution to this differential equation in terms of a product of ${}_2F_1$ hypergeometric functions. This product can be simplified into Γ's and then $\sin(x)/x$ using Gauss's hypergeomtric summation theorem. This gives a formula for the coefficient.
4. More combinatorial proofs have been given by using the shuffle algebra of iterated integrals.

"Dressed with 2's"

Theorem (BBBL, 1998)

Let $n \geq 0 \in \mathbb{Z}$, *write*

$$I = \{ \text{ all } 2n+1 \text{ ways of inserting 2 into } \{1,3\}^n \ \} \ .$$

Then

$$\sum_{\mathbf{s} \in I} \zeta(\mathbf{s}) = \frac{\pi^{4n+2}}{(4n+3)!}$$

Example

For $n = 2$, have

$$\zeta(2,1,3,1,3) + \zeta(1,2,3,1,3) + \zeta(1,3,2,1,3) +$$
$$\zeta(1,3,1,2,3) + \zeta(1,3,1,3,2) = \frac{\pi^{10}}{11!}$$

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1. Borwein, Bradley, Broadhurst and Lisonek were able to generalise some of the combinatorics of the proof of the Zagier identity to obtain a version 'dressed with 2's'. (The preprint proving Zagier's identity was published later than this.)
2. The identity is given by inserting into the gaps between the argument string $\{1,3,1,3,\ldots\}$ a single 2, in all possible ways. One gets $2n+1$ new MZV's, and the sum of these is proven to be $\pi^{4n+2}/(4n+3)!$.

Cyclic insertion conjecture

Numerical experimentation lead to conjectural generalisation.

Notation

Let $a_1, \ldots, a_{2n+1} \in \mathbb{Z}_{\geq 0}$. Write

$$Z(a_1, \ldots, a_{2n+1}) = \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

Conjecture (Cyclic insertion - BBBL, 1998)

$$\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} Z(a_{\sigma(1)}, \ldots, a_{\sigma(2n+1)}) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Shorthand: "wt" is weight of MZV's on the LHS

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1. Much numerical experimentation by these authors lead to a large conjectural generalisation of this 'dressed with 2's identity. One can see the dressed with 2's identity as a 2 in the first slot, then cyclically shifting it. This is the right direction to generalise.
2. One takes some blocks of 2's of lengths $a_1, \ldots, a_{2n+1}$. Insert $\{2\}^{a_i}$ into the i-th gap. The sum all cyclic permutations of these blocks. The result always seems to be $\pi^{\mathrm{wt}}/(\mathrm{wt}+1)$, in particular independent of the sizes of the blocks. It only depends on their sum and number.
3. It becomes rather messy to always write the precise formula for the weight of the MZV's, so I will use the shorthand 'wt' instead. Here the weight is $2(a_1 + \ldots + a_{2n+1}) + 4n$.

Special cases

- $n = 0$

$$\zeta(\{2\}^{a_1}) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad ✓$$

- $a_1 = \cdots = a_{2n+1} = 0$

$$(2n+1)\zeta(\{1,3\}^n) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad ✓$$

- $a_1 = 1, a_2 = \cdots = a_{2n+1} = 0$

Zagier-Broahurst dressed with 2's ✓

- $a_1 = \cdots = a_{2n+1} = m$

$$(2n+1)\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad ?$$

Previously conjectured by BBB (1997).

1. We can look at some special cases of this to see how plausible the conjecture is.
2. If we take a single block of size a_1, then we get the known evaluation of $\zeta(\{2\}^{a_1}$. This evaluation is proven in BBB1996 using generating functions, though one can also see proofs in Hoffman and trace it back (in principle) to Euler's evaluation of $\zeta(2)$.
3. Taking all the blocks of 2 to have size 0 gives $2n+1$ copies of $\zeta(\{1,3\}^n)$. So we recover Zagier's formula, and see an interpretation of the $2n+1$ factor.
4. Taking $a_1 = 1$, and the rest 0 gives us the Zagier identity dressed with 2's, which also is proven.
5. If now we take all $a_i = m$, then we get $2n+1$ copies of a certain MZV. This conjecture gives an evaluation of it, which matches an earlier conjecture by BBB. This conjecture has not yet been resolved.

Bowman-Bradley

Best result so far is

Theorem (Bowman-Bradley, 2002)

Let $n, t \geq 0 \in \mathbb{Z}$, *then*

$$\sum_{\substack{a_1+\cdots+a_{2n+1}=t \\ a_i \geq 0}} Z(a_1, \ldots, a_{2n+1}) = \frac{1}{2n+1} \binom{t+2n}{t} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Compatible with cyclic insertion: Any permutation of a composition $a_1 + \cdots + a_{2n+1} = t$ is still a composition.

Will use the motivic MZV framework to improve on this, up to $\mathbb{Q}$.

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1. The best result so far generalised the combinatorics of the Zagier-with-2's identity. It views this as inserting all possible blocks of 2's which have total length 1, and so generalises to all compositions of lengths.
2. This result is compatible with cyclic insertion: there are $\binom{t+2n}{t}$ compositions of t into $2n+1$ parts. We obtain all permutations of a fixed composition, so can combine them with cyclic insertion to get $\frac{\pi^{\mathrm{wt}}}{(|wt+1)!}$. This means each term contributes on average $\frac{1}{2n+1}\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$, giving the total above.
3. Later we see how to use the motivic MZV framework to improve this to a sum over a smaller index set, at the expense of getting an identity up to $\mathbb{Q}$.

Hoffman's conjecture

Separate conjecture, with a similar flavour

Conjecture (Hoffman, MZV Infopage, 2000)

For $m \geq 0 \in \mathbb{Z}$,

$$2\,\zeta(3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,2) \stackrel{?}{=} -\zeta(\{2\}^{m+3}) = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Verified up to weight 22, $m = 8$ using MZV datamine, Vermaseren (2009).

Will prove this up to $\mathbb{Q}$, using the motivic MZV framework.

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1. Now for something completely different. Hoffman lists the following conjectural identity on his homepage, as an example of identities which can be discovered with the EZ-Face engine. The identity has a rather similar flavour to the previous: zetas of 1's, 2's and 3's summing up to some multiple of $\pi^{\mathrm{wt}}/(\mathrm{wt}+1)!$. But it is different enough that there doesn't seem to be much of a connection.
2. Hoffman notes that this identity has been checked up to weight 22 using the datamine, but the general case is unproven.
3. Later we see a motivic proof of this (indeed even a generalisation), but only up to $\mathbb{Q}$

Unification and generalisation

Goal

Cyclic insertion and Hoffman are *special cases* of a more general (conjectural) family.

Can produce many new (conjectural) identities.

Example

$$\zeta(\{2\}^m, 1, 3, 3, 1, 2) + \zeta(3, 1, 2, 1, \{2\}^m, 3) - \zeta(1, 2, 1, \{2\}^m, 3, 1, 2) + \\ + \zeta(1, 2, 1, 3, 3, \{2\}^m) - \zeta(3, \{2\}^m, 1, 3, 3) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

(For above: $\in \pi^{\mathrm{wt}}\mathbb{Q}$ holds. Generally can use motivic MZV's to prove certain *symmetrised* versions, up to $\mathbb{Q}$.)

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1. I want to claim now that Hoffman's identity and the BBBL cyclic insertion conjecture come from exactly the same construction. They are both special cases of something much more general. This generalisation will allow us to write down many more (conjectural) identities with the same general flavour.
2. One such example is the above 5-term identity, which in fact holds up to $\mathbb{Q}$ using motivic MZV's.
3. We can't show the general identity holds, only some symmetrised version, but this will still be enough to improve on what is currently known for Hoffman's identity and the BBBL identity.

Motivic MZV's and algebraic tools

Motivic MZV's and algebraic tools

1. To be able to motivate, state and prove these generalisations we need to introduce some algebraic tools, namely the motivic MZV's defined by Brown.

MZV's as iterated integrals

$$\zeta(s_1,\ldots,s_r) = (-1)^r I(0;\underbrace{1,0,\ldots,0}_{s_1},\ldots,\underbrace{1,0,\ldots,0}_{s_r};1)$$

where

$$I(a_0;a_1,\ldots,a_N;a_{N+1}) = \int_{a_0\leq t_1<\cdots<t_N\leq a_{N+1}} \frac{\mathrm{d}t_1}{t_1-a_1}\cdots\frac{\mathrm{d}t_N}{t_N-a_N}$$

Convergent if $a_1 \neq a_0$ and $a_N \neq a_{N+1}$

Properties

- $I(0;a_1,\ldots,a_N;0) = 0$ for $N \geq 1$ (Equal boundaries)
- $I(a_0;a_1,\ldots,a_N;a_{N+1}) = I(1-a_0;1-a_1\ldots,1-a_N;1-a_{N+1})$ (Functoriality)
- $I(a_0;a_1,\ldots,a_N;a_{N+1}) = (-1)^N I(a_{N+1};a_N,\ldots,a_1;a_0)$ (Reversal of paths)
- $I(a;w;b)I(a;v;b) = I(a;w ⧢ v;b)$ (Shuffle product)

MZV's as iterated integrals

1. Well known formula due to Kontsevich shows how to write MZV's as iterated integrals over $0,1$, of forms dx/x and $dx/(1-x)$
2. These iterated integrals satisfy a number of properties. For us some of the more useful ones are as follows. If the two bounds of integration are equal, then the integral (obviously) vanishes. The integral satisfies functoriality under the map $t\mapsto 1-t$ (and more generally). If we reverse the path of integration, we pick up a sign.
3. Combining functionality and reversal of paths leads to the duality of MZV's: the result of reverse and $0\leftrightarrow 1$ leads to an integral of MZV type, so we get equality between pairs of MZV's, which is otherwise not easy to see.
4. Finally we have the shuffle product multiplication of these integrals. $w ⧢ v$ is the 'riffle shuffle' of w with v: all ways of interleaving the two words while preserving their original orders. By shuffling out leading $a_1 = 0$ variables, we obtain the shuffle regularisation of divergent integrals with $a_1 = 0$ or $a_N = 1$.

Brown's motivic MZV's (See Winter school)

- Algebra $\mathcal{H}$ of motivic MZV's

$$\zeta^{\mathfrak{m}}(s_1, \ldots, s_r) \coloneqq [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 \backslash \{ 0, 1, \infty \}, \vec{1}_0, -\vec{1}_1)), \overbrace{\mathrm{dch}}^{\text{straight line}}, \underbrace{\Omega}_{\text{integrand}}]^{\mathfrak{m}} .$$

  Contains all motivic iterated integrals

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}), a_i \in \{ 0, 1 \}$$

- Projection to algebra $\mathcal{A}$ of de Rham motivic MZV's

$$\zeta^{\mathfrak{a}}(s_1, \ldots, s_r) \coloneqq [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 \setminus \{ 0, 1, \infty \}, \vec{1}_0, -\vec{1}_1)), \underbrace{\varepsilon}_{\text{augmentation ideal}}, \Omega]^{\mathfrak{m}} ,$$

  kernel generated by $\zeta^{\mathfrak{m}}(2)$.

- Coaction

$$\Delta \colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$$

  lifts Goncharov's 'semicircular' coproduct on $\mathcal{A}$. $\mathcal{H}$ Hopf algebra comodule over $\mathcal{A}$.

1. Motivic MZV's are defined as certain triples, which give objects in the ring of motivic periods. Integrating the previous integrand on the straight line path in the motivic fundamental group of $\mathbb{P}^1 - \{\infty, 0, 1\}$ gives us the motivic zeta. Period map to $\mathbb{C}$.
2. Get motivic analogues of the iterated integrals here, as divergent iterated integrals can be regularised to MZV's. Motivic integrals satisfy the same nice properties as before.
3. Moreover, we have the de Rham version of this construction, which gives de Rham motivic MZV's. There is a projection map from motivic MZV's to de Rham motivic MZV's whose kernel is exactly $\zeta^{\mathfrak{m}}(2)$, so one can think of just killing $\zeta^{\mathfrak{m}}(2)$.
4. The resulting motivic MZV's naturally form a Hopf algebra comodule, so there is a coaction. This coaction is given by some lifting of Goncharov's semicircular coproduct formula. Introduce infinitesimal version of it next, so perhaps can indicate how Δ looks like.

Infinitesimal coproduct

Definition (Derivations D_k)

Let $\mathcal{L} := \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

$$\begin{aligned} D_k \colon \quad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id})\, I^{\mathfrak{m}}(w) \end{aligned}$$

$$D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) =$$

$$\sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes \qquad \rightsquigarrow \text{Subsequence}$$

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1) \qquad \rightsquigarrow \text{Quoient sequence}$$

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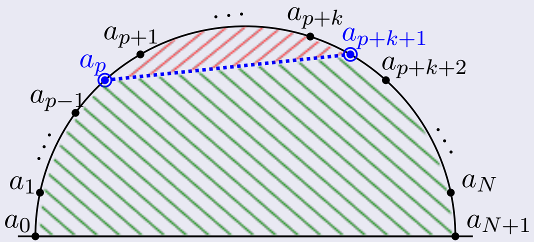
$D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) = \sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes$ ⇝ Subsequence
$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$ ⇝ Quoient sequence

1. The full coaction is given by a long formula, and is difficult to work with in general. It has order n^2 terms. Instead Brown introduces an infinitesimal (linearised) version of it which only has a linear number of terms, as follows.
2. First go to the Lie coalgebra of indecomposables $\mathcal{L}$ by killing products. Then we can project the weight k part of the coaction to this, to define the operator D_k.
3. Using the (not given) formula for Δ, one can compute how D_k acts on any integral $I^{\mathfrak{m}}(w)$, giving this explicit formula.
4. It is useful to introduce some terminology here for the two terms appearing in D_k. The term on the left is the subsequence (as the arguments form one). The term on the right is the quotient sequence, since we kill the subsequence to get it.

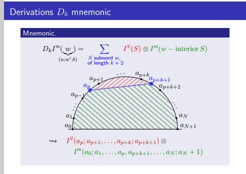
Derivations D_k mnemonic

Mnemonic.

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w, \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow \quad I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes$$
$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$$



1. A much better way to interpret/use this formula for D_k is using a pictorial mnemonic.
2. Schematically the formula for D_k is given by a sum over all subwords of w, and forming the sub/quotient sequence. If we arrange the points of w around a 'semicircular' polygon, then the terms are formed by cutting off segments containing k points. This splits the polygon into two parts: a main polygon in green, and a cut off polygon in red. From the main polygon we form the right hand term, and from the quotient polygon we form the left hand term.
3. This is the same mnemonic that can be used for Goncharov's coproduct/the full coaction: but we take all contiguous n-tuples of segments starting from a_0 and ending at a_{N+1}. Form the product of every cut-off polygon to get the left hand factor.

Transcendental Galois Theory

Theorem (Brown, 2012)

Let $D_{<N} = \bigoplus_{1<2r+1<N} D_{2r+1}$. *In weight* N,

$$\ker D_{<N} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}\,.$$

Example

Can show $\zeta^{\mathfrak{m}}(\{2\}^n) = \pm I^{\mathfrak{m}}(0; \underbrace{1,0,1,0,\ldots,1,0}_{n \text{ times}}; 1) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$

- Integral word alternates 0 and 1
- Odd length subsequence has same boundaries, vanishes
- Therefore all D_{2r+1} vanish

Conclude $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

1. The real upshot of this construction comes with the following theorem, a kind of transcendental Galois theory.
2. We can check that $\zeta(N)$ vanishes under all D_{odd} simultaneously. Brown shows that this actual characterises $\zeta^{\mathfrak{m}}(N)$. Any motivic MZV of weight N which vanishs under all D_{odd} must be a rational multiple of $\zeta(N)$. This result forms the basis of an exact-numerical algorithm for decomposing MZV's into a chosen basis.
3. For us, this result will be used to check/prove identities up to $\mathbb{Q}$. Here is a quick example, which shows that $\zeta(\{2\}^n)$ is a rational multiple of $\zeta(2n)$. (Confirming one of the known special cases of cyclic insertion.)
4. $\zeta(\{2\}^n)$ is described by the integral $I(0,(1,0)^n,1)$. Any subsequence of odd length must start and end with the same symbols since the word w has period 2. This means every subsequence vanishes trivially: the bounds of integration are the same. The result follows immediately.

$\zeta^{\mathfrak{m}}(\{1,3\}^n)$

More interesting: $\zeta^{\mathfrak{m}}(\{1,3\}^n) = I^{\mathfrak{m}}(0; (1100)^n; 1) \in \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

- Word has period 4, so length $1 \pmod 4$ subsequence vanish
- For length $3 \pmod 4$, look at starting position

$$1 \pmod 4: \quad I^{\mathfrak{L}}(0; (1100)^a 1; 1) \otimes I^{\mathfrak{m}}((0110)^b 0 \mid 10(0110)^c 01)$$
$$2 \pmod 4: \quad I^{\mathfrak{L}}(1; 1(0011)^a; 0) \otimes I^{\mathfrak{m}}((0110)^b 01 \mid 0(0110)^c 01)$$

- Cancel using reversal of paths in $I^{\mathfrak{L}}$. Similar for position $3, 4 \pmod 4$
- See cancellation as 'reversing' segments. Involution pairs up subsequences:

$$I^{\mathfrak{m}}(01 \mid 10 \mid 0\,1 \mid 10 \mid \cdots \mid 10 \mid 01 \mid 1\,0 \mid 01)$$

Conclude $\zeta^{\mathfrak{m}}(\{1,3\}^n) \in \ker D_{<4n} = \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

$\zeta^{\mathfrak{m}}(\{1,3\}^n)$
More interesting: $\zeta^{\mathfrak{m}}(\{1,3\}^n) = I^{\mathfrak{m}}(0; (1100)^n; 1) \in \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$
Word has period 4, so length $1 \pmod 4$ subsequence vanish
For length $3 \pmod 4$, look at starting position
$1 \pmod 4: I^{\mathfrak{L}}(0; (1100)^a 1; 1) \otimes I^{\mathfrak{m}}((0110)^b 0 \mid 10(0110)^c 01)$
$2 \pmod 4: I^{\mathfrak{L}}(1; 1(0011)^a; 0) \otimes I^{\mathfrak{m}}((0110)^b 01 \mid 0(0110)^c 01)$
Cancel using reversal of paths in $I^{\mathfrak{L}}$. Similar for position $3, 4 \pmod 4$
See cancellation as 'reversing' segments. Involution pairs up subsequences:
$I^{\mathfrak{m}}(01 \mid 10 \mid 0\,1 \mid 10 \mid \cdots \mid 10 \mid 01 \mid 1\,0 \mid 01)$
Conclude $\zeta^{\mathfrak{m}}(\{1,3\}^n) \in \ker D_{<4n} = \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

1. A more interesting case is $\zeta(\{1,3\}^n)$, the Zagier-Broadhurst identity. This time the word w is $0(1100)^n 1$, periodic with period 4.
2. If the subsequence has length $1 \pmod 4$, then the we see that the subsequence again trivially vanishes. However if the subsequence has length $3 \pmod 4$, things don't just vanish, and we have to look at where the subsequence starts.
3. If the subsequence starts at position $1 \pmod 4$, we get a contribute to D_k which looks as indicated. And starting at $2 \pmod 4$ gives the other term.
4. Using reversal of paths, we can revere the second $I^{\mathfrak{L}}$, shows the two terms are equal up to a -1. So they cancel in D_k. Something similar happens for terms starting $3, 4 \pmod 4$.
5. In some sense we can see this cancellation as a pairing up of terms indicted by a reversal/reflection of segments. Either way, this shows all terms in D_k cancel, and by Brown we get the result. We are going to generalise this cancellation observation.

Alternating block decomposition

Alternating block decomposition

Alternating blocks

Observation

In $\zeta^{\mathfrak{m}}(\{1,3\}^n)$ proof, points 00 and 11 in w are 'somehow' significant.

- Splitting here decomposes a word into *alternating blocks* $0101\cdots$ or $1010\cdots$.

Definition (Block decomposition)

Let w be a word starting with $\varepsilon_1 \in \{0,1\}$. Write w as alternating blocks, with lengths $\ell_1, \ldots, \ell_k$. The block decomposition of w is

$$\mathrm{bl}(w) = (\varepsilon_1; \ell_1, \ldots, \ell_k)\,.$$

Example

$$\mathrm{bl}(\underbrace{0}_{1} \mid \underbrace{01}_{2} \mid \underbrace{10}_{2} \mid \underbrace{01010}_{5} \mid \underbrace{0}_{1} \mid \underbrace{01}_{2}) = (0; 1, 2, 2, 5, 1, 2)$$

1. Somehow the 00 and 11 points determined the cancellation pairing i the $\zeta(\{1,3\}^n)$ proof, so they play a significant role. When cut at these points, we get segments of the form 01 or 10, which consist of alternating 0's and 1's. Let's do this in general.
2. Cut any word w at points 00 and 11, and we get an expression for w as a concatenation of alternating blocks 010101 and 1010101. Make a note of the lengths of these blocks, and this defines the block decomposition of w, as in the example. In order to recover w from this construction, we definitely need to know the starting point, so this is part of the block decomposition too.

Alternating blocks

Can recover w from $(\varepsilon_1; \ell_1, \ldots, \ell_k)$: blocks arise from $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$.

Notation

Write $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k) = I(\mathrm{bl}^{-1}(\varepsilon_1; \ell_1, \ldots, \ell_k))$. If $\varepsilon_1 = 0$, just write $(\ell_1, \ldots, \ell_k)$.

- Weight of $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k)$ is $-2 + \sum_i \ell_i$. (Bounds of integration are counted!)
- If $\mathrm{wt} \equiv k \pmod 2$ then $I_{\mathrm{bl}} = 0$. (End points are equal!)
- I_{bl} is divergent iff $\ell_1 = 1$ or $\ell_k = 1$.

Example

$$I_{\mathrm{bl}}(1, 2, 2, 5, 1, 2) = I(0; 01100101000; 1)$$

Alternating blocks
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Example
$I_{\mathrm{bl}}(1, 2, 2, 5, 1, 2) = I(0; 01100101000; 1)$

1. But once we known the starting point and the block lengths, it is easy to see how to recover w from them. The blocks arise from cutting $00 \to 0 \mid 0$ and $11 \to 1 \mid 1$. So consecutive blocks end/start with the same symbol. If we know where to start we can just write down blocks of the appropriate lengths to get w.
2. This means bl is invertible, and from this we can define the block integral: it is just the integral given by the word with corresponding block decomposition.
3. Some properties of this: the weight of the integral is $\sum \ell_i$ minus 2, because we also count the end points of integration in the block decomposition.
4. If the weight and number of blocks are the same mod 2, then the integral is trivially 0 because the end points are equal. We integrate from 0 to 0, or from 1 to 1.
5. Divergent integrals start with 00 or end with 11, so integrals are divergent iff the first block has $\ell_1 = 1$, or the last has $\ell_k = 1$.

Block structure of BBBL conjecture

- Write the BBBL identity as iterated integrals

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

$$\rightsquigarrow \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1}1(10)^{a_2}100\cdots01(10)^{a_{2n}}100(10)^{a_{2n+1}}1)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1}1 \mid (10)^{a_2}10 \mid 0\cdots01 \mid (10)^{a_{2n}}10 \mid 0(10)^{a_{2n+1}}1)$$

- Record lengths of the blocks

$$= \pm \sum_{\text{cycle } a_i} I_{\text{bl}}(2a_1+2, 2a_2+2, \ldots, 2a_{2n+1}+2)$$

- Right hand side is $\zeta(\{2\}^{\text{wt}/2}) = \pm I_{\text{bl}}(\text{wt}+2)$.

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1. Now let us apply this construction to gain a new understand the structure of the BBBL identity
2. When converting to integrals we have a sign corresponding to the depth. For the left hand side the sign is constantly $(-1)^(2n + \sum a_i) = (-1)^(\text{wt}/2)$.
3. Each term gives an integral with even sized blocks $2a_i + 2$.
4. The right hand side of $\zeta(\{2\}^{\text{wt}/2})$, so again we pick up sign $(-1)^(\text{wt}/2)$.
5. So in the resulting integral identity, we can cancel the sign to make every term positive. We get the structure: sum over cyclic shifts of block lengths is block of weight+2.

Block structure of Hoffman's conjecture

- Write Hoffman's identity as iterated integrals

$$\begin{aligned} & 2\zeta(3,3,\{2\}^n) && -\zeta(3,\{2\}^n,1,2) && \\ = {} & \zeta(3,3,\{2\}^n) && -\zeta(3,\{2\}^n,1,2) && +\zeta(\{2\}^n,1,2,1,2) \\ \rightsquigarrow {} & \pm\big(I(0100100(10)^n1) + I(0100(10)^n1101) + I(0(10)^n1101101)\big) \end{aligned}$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$\begin{aligned} = \pm\big(& I(010 \mid 010 \mid 0(10)^n1) + I(010 \mid 0(10)^n1 \mid 101) \\ & + I(0(10)^n1 \mid 101 \mid 101)\big) \end{aligned}$$

- Record lengths of the blocks

$$= \pm\big(I_{\mathrm{bl}}(3,3,2n+2) + I_{\mathrm{bl}}(3,2n+2,3) + I_{\mathrm{bl}}(2n+2,3,3)\big)$$

- Right hand side is $-\zeta(\{2\}^{n+3}) = \pm I_{\mathrm{bl}}(\mathrm{wt}+2)$

Block structure of Hoffman's conjecture
Write Hoffman's identity as iterated integrals
Split into 'alternating blocks' at 00 → 0 | 0 or 11 → 1 | 1
Record lengths of the blocks
Right hand side is −ζ({2}^{n+3}) = ±I_bl(wt + 2)

1. Now let us apply this construction to gain a new understand the sturcture of the Hoffman identity
2. We a first step, I want to split up the coefficient 2 term using duality. Maybe this is unmotivated currently, but it is useful in a moment.
3. When converting to integrals we have a sign corresponding to the depth. For the left hand side the sign is overall $(-1)^(n)$, as some terms have one extra argument, and one extra minus sign.
4. Each term gives block of size $3, 3, 2n+2$ in some order. Surprising?
5. The right hand side of $-\zeta(\{2\}^{n+3})$, so again we pick up sign $-(-1)^{n+3} = (-1)^(n)$.
6. So in the resulting integral identity, we can cancel the sign to make every term positive. We get the structure: sum over cyclic shifts of block lengths is block of weight+2.

Common structure and generalisation

Both conjectures have same structure: cyclic permutations of block lengths ℓ_i.

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ *with all* $\ell_i > 1$,

$$\sum_{\text{cycle } \ell_i} I_{\text{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\text{bl}}(\text{wt} + 2) = \begin{cases} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!} & \text{wt } \textit{even} \\ 0 & \text{wt } \textit{odd} \end{cases}$$

- Numerically tested all cases weight ≤ 18, to 500 decimal places
- Can prove a symmetrised version, up to $\mathbb{Q}$
- Can prove *some* special cases, up to $\mathbb{Q}$

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Numerically tested all cases weight ≤ 18, to 500 decimal places
Can prove a symmetrised version, up to $\mathbb{Q}$
Can prove some special cases, up to $\mathbb{Q}$

1. We see that both identities have exactly the same structure, so hopefully it is not too much of a leap to think a generalisation like this may hold.
2. Of course, we need more than just two conjectural cases to justify this pattern. So I justify this conjecture by noting that I have checked it for all cases up to weight 18, to 500 decimal places. (Could also use the datamine.) Moreover, I can prove a symmetrisation of this holds up to $\mathbb{Q}$ using motivic MZV's, and can even prove special cases on the nose, up to $\mathbb{Q}$.
3. We can also produce new candidate identities that we can individually check to very high weight numerically.

Examples

Example

Let $(\ell_1, \ldots, \ell_k) = (2m+2, 2, 3, 2, 3)$, then we obtain

$$\zeta(\{2\}^m, 1, 3, 3, 1, 2) + \zeta(3, 1, 2, 1, \{2\}^m, 3) - \zeta(1, 2, 1, \{2\}^m, 3, 1, 2) + \\ + \zeta(1, 2, 1, 3, 3, \{2\}^m) - \zeta(3, \{2\}^m, 1, 3, 3) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Proposition (C., 2017, arXiv 1703.03784)

The above identity holds up to $\mathbb{Q}$

Proof (Sketch).

Lift the identity to $\zeta^{\mathfrak{m}}$, and compute $D_{<2m+10}$. A (tedious) calculation shows $D_{<2m+10}$ vanishes. □

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Let $(\ell_1, \ldots, \ell_k) = (2m+2, 2, 3, 2, 3)$, then we obtain
$\zeta(\{2\}^m, 1, 3, 3, 1, 2) + \zeta(3, 1, 2, 1, \{2\}^m, 3) - \zeta(1, 2, 1, \{2\}^m, 3, 1, 2) + \zeta(1, 2, 1, 3, 3, \{2\}^m) - \zeta(3, \{2\}^m, 1, 3, 3) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$
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The above identity holds up to $\mathbb{Q}$
Proof (Sketch).
Lift the identity to $\zeta^{\mathfrak{m}}$, and compute $D_{<2m+10}$. A (tedious) calculation shows $D_{<2m+10}$ vanishes. □

1. If we start with the indicated block lengths, we produce this candidate identity. One can verify it numerically up to weight 100 or so, using gp/pari
2. Moreover, I claim that one can give a motivic proof to show the result holds up to $\mathbb{Q}$. However this is by a very tedious calculation writing down every single term in $D_{<N}$, and showing explicitly that they cancel. There don't seem to be any nice properties to exploit to simplify this.

Progress and results

Theorem (Symmetric insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$, *with even weight,*

$$\sum_{\text{permute } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \in I_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$$

(Odd weight holds trivially, by duality)

Proof (Strategy).

- Lift to motivic version $I^{\mathfrak{m}}$.
- Define a reflection $\mathcal{R}$ on non-trivial subsequences
- Use $\mathcal{R}$ to cancel terms in $D_{<N}$
- Conclude $\in \zeta^{\mathfrak{m}}(\mathrm{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$ using Brown.

Progress and results
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1. The general result is the following. If one sums over all permutations of the block lengths, then one is guaranteed to get are rational multiple of $I_{\mathrm{bl}}(\mathrm{wt} + 2)$. This is still far away from the cyclic insertion conjecture generally, but as we'll see in a moment already produces non-trivial new results.
2. One remark though: in the odd weight case this symmetrisation will trivially vanish, since duality reverses the block lengths. This means the permutation which reverses the block lengths gives the dual term, which we pick up an extra minus sign from the when converting back to MZV's.
3. The strategy of the proof is to use the motivic MZV framework, and generalise the $\zeta(\{1,3\}^n)$ proof. We lift this to a motivic version, and show that $D_{<N}$ vanishes by setting up a cancellation of terms.

Progress and results

Proof (Details).

$$\mathcal{R}\colon I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_s}_{\text{start at position } \alpha}, \ldots, \underbrace{\ell_t}_{\text{end at position } \beta}}^{\text{non-trivial subsequence } S}, \ldots, \ell_k)$$

$$\mapsto I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_t}_{\text{start at position } \beta}, \ldots, \underbrace{\ell_s}_{\text{end at position } \alpha}}^{\text{reflection } \mathcal{R}S}, \ldots, \ell_k)$$

- Get permutation of ℓ_i.
- Both quotients are $I^{\mathfrak{L}}_{\mathrm{bl}}(\ell_1, \ldots, \ell_{s-1}, \alpha + \beta, \ell_{t+1}, \ldots, \ell_k)$
- Subsequences are
  $I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon; \ell_s - \alpha, \ell_{s+1}, \ldots, \ell_{t-1}, \ell_t - \beta)$, and
  $I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon'; \ell_t - \beta, \ell_{t-1}, \ldots, \ell_{s+1}, \ell_s - \alpha)$
- Reverses or duals, differ by $(-1)^{\text{length}} = -1$. Cancel in $D_{<N}$ □

1. More precisely, $\mathcal{R}$ takes a non-trivial subsequence which starts at position α in block ℓ_s, and ends at position β from the end block ℓ_t. It maps this to a subsequence on another integral by reversing the blocks, and carrying the subsequence with it. (It starts and ends in the same blocks, but now the lengths ℓ_i are some permutation. However, the start and end positions are changed.)
2. We now see how the quotient and subsequences compare.
3. The quotient sequences are exactly equal. Outside of S the blocks are identical. Once we cut S out from the sequence, these blocks are joined by an alternating sequence 01010 of length $\alpha + \beta$. (Why alternating? Well the two end points of S are different, so when we just from the start to the end we go $0 \to 1$ or $1 \to 0$.)
4. Finally the subsequences? Well it is clear that the block lengths are reversed, so it depends on only the starting letter of the subsequence. If end of ℓ_t is start ℓ_s, we get reverse. Otherwise we have $0 \mapsto 1$, and get dual.

Corollaries of symmetric insertion

Corollary (Generalisation of Hoffman, up to $\mathbb{Q}$)

For $(\ell_1, \ell_2, \ell_3) = (2a+3, 2b+3, 2c+2)$, *we obtain*

$$\mathrm{Sym}_{a,b} \left(\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) \right. \\ \left. + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) \right) \in \pi^{\mathrm{wt}} \mathbb{Q}$$

Duality shows cyclic insertion already holds up to $\mathbb{Q}$

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) \\ + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) \in \pi^{\mathrm{wt}} \mathbb{Q}$$

In particular, $a = b = 0$ is Hoffman's identity up to $\mathbb{Q}$.

1. We can take the following block lengths generalising Hoffman's case, and immediately conclude the sum of these 6 MZV's is a rational multiple of π^{wt}.
2. In fact duality allows us to combine pairs of terms $\zeta(\{2\}^a, 3, \{2\}^b, 3\{2\}^c) = \zeta(\{2\}^c, 1, 2, \{2\}^b, 1, 2, \{2\}^a)$, etc to get a 3-term identity which is already cyclic insertion in this case.
3. Further specialising to $a = b = 0$ gives Hoffman's identity up to $\mathbb{Q}$.

Corollaries of symmetric insertion

Corollary (Improvement of Bowman-Bradley, up to $\mathbb{Q}$)

For $\ell_i = 2a_i + 2$, obtain

$$\sum_{\text{permute } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

"Only need permutations of a single composition."

In particular, for $a_1 = \cdots = a_n = m$

Corollary (Evaluable MZV)

The following MZV is evaluable

$$\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

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$$\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

1. If we apply this to the BBBL case, we need only to sum over all permutations of the block lengths. This gives an improvement over the Bowman-Bradley theorem where all compositions were needed. The unfortunate thing is this is a non-explicit version: I can't find the rational coefficient exactly.
2. Nevertheless, putting all the $a_i = m$ shows that the previously conjectural MZV evauation does hold, again up to $\mathbb{Q}$. This MZV is definitely some rational multiple of π^{wt}.

Further progress?

Complete motivic proof of cyclic insertion is not (yet?) possible

- Cyclic insertion has a stability under D_k
- Odd weight implies $D_{<N}(\text{even weight}) = 0$
- Problem: Must fix rational multiple of $\zeta^{\mathfrak{m}}(\mathrm{wt})$ somehow $\rightsquigarrow$ analytically or numerically...
- $D_{<N}(\text{odd weight})$ involves $I^{\mathfrak{L}}$ explicitly

$$D_7 \sum\nolimits_{\text{cycle}} I^{\mathfrak{m}}_{\mathrm{bl}}(2,10,3,2) =$$
$$\underbrace{\left(I^{\mathfrak{L}}_{\mathrm{bl}}(6,3) + I^{\mathfrak{L}}_{\mathrm{bl}}(3,3,2,1) + I^{\mathfrak{L}}_{\mathrm{bl}}(2,3,2,3) + I^{\mathfrak{L}}_{\mathrm{bl}}(1,2,2,4)\right)}_{-\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(2,3) - 2\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(3,2) + 2\zeta^{\mathfrak{L}}(3)\zeta^{\mathfrak{L}}(2,2) = 0} \otimes I^{\mathfrak{m}}_{\mathrm{bl}}(10)$$

- In general only have

$$\text{odd weight} = \sum\nolimits_k \alpha_k \zeta(2k+1)\zeta(\{2\}^{\mathrm{wt}/2-k}), \quad \alpha_k \in \mathbb{Q}$$

Further progress?
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In general only have
odd weight $= \sum_k \alpha_k \zeta(2k+1)\zeta(\{2\}^{\mathrm{wt}/2-k})$, $\alpha_k \in \mathbb{Q}$

1. Unlikely that one can prove this hold result purely motivically. However, cyclic insertion has stability under D_k. Computation of D_k reduces to $I^{\mathfrak{L}}(\cdot) \times$ cyclic insertion, so hope of a recursive/inductive partial proof.
2. In particular, odd weight holds implies D_k even vanishes, so that we know the even weight case holds up to a rational if all lower weight odd caes hold. Unfortunately the rational is not visible motivically: need a numerical evaluation, or an exact formula to continue.
3. Moreover, the odd weight case has more problems. Computing D_k odd weight leads to explicit computations of $I^{\mathfrak{L}}$. For example the following. We need to recognise the factor is a sum of products and $\zeta(2)$'s in order to see $D_7 = 0$.
4. However, one can say from the special form of D_k odd weight (namely only $\zeta(2)^l$ appears on the right), that the odd weight identity can be expressed as a sum $\zeta(odd)\zeta(\{2\}^k)$, with rational coefficients.

Recent work

Using iterated integrals over $\mathbb{P}^1 \setminus \{ \infty, 0, 1, z \}$ gives

Theorem (Hirose-Sato, 2017, arXiv 1704.06478)

The generalisation of Hoffman's identity holds exactly

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) \\ + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) = -\zeta(\{2\}^{a+b+c+3})$$

After the break:

- a further generalisation of cyclic insertion, and
- exact proofs!

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After the break:
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exact proofs!

1. A more optimistic approach comes from recent work by Hirose and Sato, where they use iterated integrals over $\mathbb{P}^1 \setminus \{ \infty, 0, 1, z \}$ to give an exact proof of the generalised version of Hoffman's identity. Computer assistance was needed to find the right combination of integrals to work with, but then the proof is a simple exercise in computing the derivative.
2. Moreover, they are able to generalise the cyclic insertion conjecture to a 'block-shuffle' identity, and give a proof of this more general result.

Extra material (time permitting)

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Full version of cyclic insertion

If some $\ell_i = 1$, the identity involves product term corrections.

$$\mathcal{L}_d = \{ \, (m_{d+1}, \ldots, m_k) \mid (\overbrace{1, \ldots, 1}^{d \text{ times}}, m_{d+1}, \ldots, m_k) \text{ is a cyclic permutation of } (\ell_1, \ldots, \ell_k) \, \}$$

"Take all cyclic permutations of $(\ell_1, \ldots, \ell_k)$ which start with d consecutive 1's. Then drop the initial 1's"

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ of weight N,

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\mathrm{bl}}(N+2) - \sum_{d=1}^{\lfloor k/2 \rfloor} \frac{2(2\pi \mathrm{i})^{2d}}{(2d+2)!} \sum_{\mathbf{m} \in \mathcal{L}_{2d}} I_{\mathrm{bl}}(\mathbf{m}) \, .$$

1. The version of cyclic insertion I stated above required that all the $\ell_i > 1$, so that there was no shuffle-regularisation occurring. If any of the $\ell_i = 1$, then eventually it is cycled into the first (and/or last) position, and so the resulting integral is divergent and needs to be regularised.
2. One can regularise the resulting integrals, and give a generalisation of the identity which includes product term corrections.
3. These product terms are obtained by dropping the increasingly long divergent sequences $1, 1, \ldots, 1$ and complementing this with powers of π. One could perhaps say that cyclic insertion holds modulo π^2?

Full version of cyclic insertion

Example

With $(\ell_i) = (1,1,2,3)$, need only $\mathcal{L}_2 = \{\, (2,3) \,\}$. Get

$$I_{\mathrm{bl}}(1,1,2,3) + I_{\mathrm{bl}}(1,2,3,1) + I_{\mathrm{bl}}(2,3,1,1) + I_{\mathrm{bl}}(3,1,1,2)$$

$$\overset{?}{=} I_{\mathrm{bl}}(7) - \frac{2(2\pi \mathrm{i})^2}{4!} I_{\mathrm{bl}}(2,3)$$

Shuffle regularisation gives

$$\begin{aligned}&(3\zeta(1,1,3) + 2\zeta(1,2,2) + \zeta(2,1,2)) + \\ &(\zeta(2,3) - 6\zeta(1,1,3) - 4\zeta(1,2,2) - 2\zeta(2,1,2)) + \\ &(6\zeta(1,1,1,2)) + (-\zeta(5)) \overset{?}{=} 0 + \zeta(2)\zeta(1,2) \quad \checkmark\end{aligned}$$

Full version of cyclic insertion
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$(6\zeta(1,1,1,2)) + (-\zeta(5)) \overset{?}{=} 0 + \zeta(2)\zeta(1,2)$ ✓

1. We should, in principle compute $\mathcal{L}_4$ also, but since there is no $1,1,1,1$, subsequence it is empty.
2. The resulting block integral identity is given here. Shuffle regularising each divergent term gives the corresponding identity on MZV's which can be checked using the datamine. The order of the terms is preserved between the two lines, so one can match up the divergent integral with its shuffle regularisation.

Another block decomposition conjecture

Conjecture (BBBL 1998, rewritten)

Let $a_1, a_2, a_3, b_1, b_2 \in \mathbb{Z}_{\geq 0}$. *Then*

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \zeta(\{2\}^{a_{\sigma(1)}}, 1, \{2\}^{b_1}, 3, \{2\}^{a_{\sigma(2)}}, 1, \{2\}^{b_2}, 3, \{2\}^{a_{\sigma(3)}}) \overset{?}{=} 0$$

Generalising the block decomposition structure leads to

Conjecture (Alt-odd, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_{2k+1})$ *of even weight, with all* $\ell_i > 1$,

$$\operatorname{Alt}_{\{\, \ell_i \mid i \text{ odd} \,\}} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_{2k+1}) \overset{?}{=} 0$$

"Alternating sum over odd-position blocks."

Remark

This conjecture is included in Hirose-Sato's generalisation too.

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This conjecture is included in Hirose-Sato's generalisation too.

1. Another block decomposition conjecture comes from generalising another conjecture from BBBL, which has not received much attention. They write their identity as a sum over the dihedral group D_3, which obscures the real structure and generalisation. One one notes that it can be written as an alternating sum over S_3 instead, a generalisation readily follows.
2. Have numerically checked this for weight ≤ 18, though I don't have any motivic proofs of this. Computing D_k seems to lead to messier and messier identities at lower weight, so this result is not stable under D_k. In particular, I don't have a good odd weight version of the conjecture.

Another block decomposition conjecture

Example

For block lengths $\ell_i = 2a_i + 2$, $1 \leq i \leq 7$, get

$$\mathrm{Alt}_{a_1,a_3,a_5,a_7}\, \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \{2\}^{a_3}, 1, \{2\}^{a_4}, 3, \{2\}^{a_5}, 1, \{2\}^{a_6}, 3, \{2\}^{a_7}) \overset{?}{=} 0$$

Example

For block lengths $(2a_1 + 3, 2a_2 + 3, 2a_3 + 3, 2a_4 + 2, 2a_5 + 3)$, get

$$\mathrm{Alt}_{a_1,a_3,a_5}\, \zeta(\{2\}^{a_1}, 3, \{2\}^{a_2}, 3, \{2\}^{a_3}, 3, \{2\}^{a_4}, 1, 2, \{2\}^{a_5}) \overset{?}{=} 0$$

Another block decomposition conjecture

1. Immediately one gets the 'level' 3 generalisation to $\zeta(\{1,3\}^3)$ of the BBBL conjecture. Or one can apply it to a Hoffman identity type MZV, to get similar results. Can check each numerically to very high precision in various cases, so the conjecture looks promising.

Analogue for Multiple Zeta Star Values

Definition (MZSV)

$$\zeta^\star(s_1, s_2, \ldots, s_k) := \sum_{0<n_1\leq n_2\leq\cdots\leq n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

Theorem (Yamamoto 2013, Conjectured by ITTW 2013)

$$\sum_{\sigma\in S_{2n}} \zeta^\star(1, \{2\}^{a_{\sigma(1)}}, 3, \{2\}^{a_{\sigma(2)}}, \ldots, 1, \{2\}^{a_{\sigma(2n-1)}}, 3, \{2\}^{a_{\sigma(2n)}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

$$\sum_{\sigma\in S_{2n+1}} \zeta^\star(\{2\}^{a_{\sigma(1)}+1}, 1, \{2\}^{a_{\sigma(2)}}, 3, \{2\}^{a_{\sigma(3)}}, \ldots, 1, \{2\}^{a_{\sigma(2n)}}, 3, \{2\}^{a_{\sigma(2n+1)}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

1. One can find some analogues of these identities for MZSV's. Recall an MZSV is defined by taking $\leq$ instead of $<$ in the series.
2. Two conjectures by Imatomi, Tanaka, Tasaka, Wakabayashi suggest results similar to cyclic insertion/symmetric insertion hold for MZSV's. These results are the analogues of the original BBBL cyclic insertion conjecture.
3. Notice however, the first block of 2 is either empty, or has length $+1$.

Analogue for MZSV's

Theorem (C., 2018)

For $\ell_i > 1$,

$$\sum_{\text{permute } \ell_i} \zeta^\star(\mathrm{bl}^{-1}(2 \circ \ell_1, \ell_2, \ldots, \ell_n)) = \sum_{\substack{\mathbf{r} \in \\ \mathrm{Part}_{\mathrm{odd}}(n)}} 2^{\#\mathbf{r}} \prod_i (\# r_i - 1)! \widehat{\zeta}\Big(\sum_{j \in r_i} \ell_j\Big)$$

Where

- $\zeta^\star(0\underbrace{10\cdots0}_{s_1}\cdots\underbrace{10\cdots0}_{s_k}1) = \zeta^\star(s_1, \ldots, s_k)$
- $\circ = \begin{cases} + & \mathrm{wt} \not\equiv k \pmod 2 \\ , & \mathrm{wt} \equiv k \pmod 2 \end{cases}$ *and* $\widehat{\zeta}(s) = \begin{cases} \zeta(s) & s \text{ odd} \\ \frac{1}{2}\zeta^\star(\{2\}^{s/2}) & s \text{ even} \end{cases}$
- $\mathrm{Part}_{\mathrm{odd}}(n) = \{$ *partitions of* $\{1, \ldots, n\}$ *into odd size parts* $\}$

"A polynomial in Riemann Zeta Values."

1. It seems like the following is the 'spiritual' analogue of cyclic insertion for MSZV's. Taking a $\zeta^\star$ described using block lengths, one form the sum of all permutations of the ℓ_i, and one can write the result in an explicit way.
2. The behaviour of the first block of 2's (either empty or length $+1$) is governed by the $\circ =$, or $+$. For the case all ℓ_i even, we have all the sums of ℓ_i on the RHS are even, so the zeta is a power of π. Hence the entire RHS is a rational times π^{wt}.
3. Some curious features though: this resembles the symmetric sum formula (not a coincidence), but involves only odd sized partitions. Moreover, the block lengths on the LHS become arguments on the RHS. Also not a coincidence. This is explained in the proof.

Analogue for MZSV's - Proof

Proof (Sketch).

- Apply Zhao's (generalised) 2-1 formula

$$\zeta^\star(\mathbf{s}) = \varepsilon(\mathbf{s}) \sum_{\mathbf{p} \in \Pi(\mathbf{s}^{(1)})} 2^{\#\mathbf{p}} \zeta(\mathbf{p})$$

- Show $\mathbf{s}^{(1)} = (\widetilde{\ell}_1, \ldots, \widetilde{\ell}_k)$ where

$$\widetilde{\ell}_j = \begin{cases} \ell_j & \ell_j \text{ odd} \\ \overline{\ell_j} & \ell_j \text{ even} \end{cases} \quad \rightsquigarrow \text{Alternating MZV's}$$

- Apply (Zhao's generalisation of) the symmetric sum formula
- Use Zobilin's evaluation

$$\zeta(\overline{2n}) = -\tfrac{1}{2}\zeta^\star(\{2\}^n)$$

□

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$\zeta^\star(\mathbf{s}) = \varepsilon(\mathbf{s}) \sum_{\mathbf{p} \in \Pi(\mathbf{s}^{(1)})} 2^{\#\mathbf{p}} \zeta(\mathbf{p})$
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$\widetilde{\ell}_j = \begin{cases} \ell_j & \ell_j \text{ odd} \\ \overline{\ell_j} & \ell_j \text{ even} \end{cases}$ ⇝ Alternating MZV's
Apply (Zhao's generalisation of) the symmetric sum formula
Use Zobilin's evaluation
$\zeta(\overline{2n}) = -\frac{1}{2}\zeta^\star(\{2\}^n)$

1. Zhao's generalised 2-1 formula evaluates a $\zeta^\star$ as a certain sum of alternating MZV's, where $\mathbf{s}^{(1)}$ is some recursively defined index string $(a_1 \circ \cdots \circ a_k)$, and re have $\circ = \oplus$ or , in all possible ways.
2. The surprising observation is that this index string is almost exactly the block decomposition of the MZV. The initial 2 is lost, and the other block lengths become $\overline{\ell_i}$ if they are even.
3. This reduces the result to a sum of symmetric sums of alternating MZV's which can be explicitly evaluated using the (generalisation) of Hoffman's symmetric sum formula.
4. Finally one converts the $\zeta(\overline{2n})$'s back to $\zeta^\star$'s using Zobilin's evaluation (which is a special case of Zhao).
5. The existence of some formula like the Zhao 2-1 result, but for usual zetas, could lead to an explicit proof of cyclic insertion.

Analogue for MZSV's - Example

Example (Hoffman analogue)

For $(\ell_i) = (2a+3, 2b+3, 2c+2)$, have $\circ = +$, and

$$\mathrm{Part}_{\mathrm{odd}}(3) = \{\, \{1 \mid 2 \mid 3\}, \quad \{123\} \,\} \,.$$

Obtain

$$\begin{aligned}
&\zeta^\star(\{2\}^{a+1}, 3, \{2\}^b, 3, \{2\}^c) \qquad + \zeta^\star(\{2\}^{b+1}, 3, \{2\}^a, 3, \{2\}^c) + \\
&+ \zeta^\star(\{2\}^{b+1}, 3, \{2\}^c, 1, 2, \{2\}^a) \; + \zeta^\star(\{2\}^{a+1}, 3, \{2\}^c, 1, 2, \{2\}^b) + \\
&+ \zeta^\star(\{2\}^{c+1}, 1, 2, \{2\}^a, 1, 2, \{2\}^b) + \zeta^\star(\{2\}^{c+1}, 1, 2, \{2\}^a, 1, 2, \{2\}^b)
\end{aligned}$$

$$\begin{aligned}
&= 2^3 (1-1)!^3 \zeta(2a+3)\zeta(2b+3) \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{c+1}) + && \rightsquigarrow \mathbf{r} = \{1 \mid 2 \mid 3\} \\
&\qquad + 2^1 (3-1)! \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{a+b+c+4}) && \rightsquigarrow \mathbf{r} = \{123\}
\end{aligned}$$

$$= 4\zeta(2a+3)\zeta(2b+3)\zeta^\star(\{2\}^{c+1}) + 2\zeta^\star(\{2\}^{a+b+c+4})$$

1. So for example, starting with the Hoffman block lengths, one obtains some kind of MZSV analogue of Hoffman's identity.

Summary

- Defined block decomposition of an iterated integral
- Used block decomposition to unify/generalise BBBL and Hoffman's conjectures
- Used motivic MZV's to prove a symmetrised version holds
- Improved Bowman-Bradley to only permutations, proved Hoffman, and other identities up to $\mathbb{Q}$

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