Eta quotients, Eichler integrals and L-series

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Physicists have recently come to appreciate the utility of Eichler integrals of quotients of the Dedekind eta function. In this survey, I focus on some of the mathematical features that are involved, including modular transformations of the higher normal functions investigated by Bloch, Kerr and Vanhove, critical and non-critical values of L-series, quasi-periods in the sense of Brown and Hain, and quadratic relations between periods encoded by Betti and de Rham matrices. No familiarity with physical context will be assumed. In many places I rely on inspired proofs recently devised by Yajun Zhou.

1. Eta quotients at sunrise: optimal use of modular transformations.
2. Three-loop sunrise: improving the result of Bloch, Kerr and Vanhove.
3. Eichler integrals, on shell, up to six loops, with help from Anton Mellit.
4. A link to Francis Brown's study of quasi-periods on $\Gamma_0(6)$.
5. Critical L-series up to 22 loops, from work with David Roberts.
6. Quadratic relations between Feynman periods for all loops.

1 Eta quotients at sunrise

With $q := \exp(2\pi \mathrm{i} z)$ and $\Im(z) > 0$, the **Dedekind eta** function satisfies

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24} = \frac{\eta(-1/z)}{\sqrt{-\mathrm{i}z}}.$$

With $\eta_n := \eta(nz)$ and $\chi_3(n) = 0, 1, -1$, for $n = 0, 1, 2 \bmod 3$, I expand

$$\begin{aligned}
w &:= 3\,\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2} = 3\left(1+4q+12q^2+28q^3+60q^4+120q^5\right)+O(q^6),\\
\frac{w^2-1}{8} &= \frac{\eta_2^9\eta_3^3}{\eta_1^9\eta_6^3} = 1+9q+45q^2+171q^3+549q^4+1566q^5+O(q^6),\\
\frac{w^2-9}{72} &= \frac{\eta_2\eta_6^5}{\eta_1^5\eta_3} = q+5q^2+19q^3+61q^4+174q^5+O(q^6),\\
f &:= \frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2} = 1-6\sum_{n=1}^{\infty}\frac{\chi_3(n)q^n}{1+q^n},\ \text{at weight 1},\\
g &:= \frac{\eta_2^5\eta_3^4\eta_6}{\eta_1^4} = \frac{\eta_3^9}{\eta_1^3}+\frac{\eta_6^9}{\eta_2^3} = \sum_{n=1}^{\infty}\frac{n^2(q^n-q^{5n})}{1-q^{6n}},\ \text{at weight 3, for 2 loops},\\
h &:= \frac{\eta_2^{16}}{\eta_1^8}-9\frac{\eta_6^{16}}{\eta_3^8} = \sum_{n=1}^{\infty}\frac{n^3(q^n-8q^{3n}+q^{5n})}{1-q^{6n}},\ \text{at weight 4, for 3 loops}.
\end{aligned}$$

Broadhurst, **Fleischer** and **Tarasov** (BFT) gave the differential equation for the D-dimensional unit-mass **sunrise** integral. At $D = 2$, this yields

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{I(w^2)}{6f} = g, \quad I(w^2) = 4\int_0^\infty I_0(wx)K_0^3(x)x\mathrm{d}x,$$

with w^2 the norm of the external momentum. The elliptic **periods**

$$f = \frac{4\sqrt{3}}{\mathbf{agm}\left(\sqrt{(w+3)(w-1)^3}, \sqrt{16w}\right)} = \frac{\sqrt{3}\Im I(w^2+\mathrm{i}0)}{\pi^2} \text{ for } w > 3,$$

$$2zf = \frac{4\sqrt{-3}}{\mathbf{agm}\left(\sqrt{(w+3)(w-1)^3}, \sqrt{(w-3)(w+1)^3}\right)},$$

yield the **nome** $q := \exp(2\pi\mathrm{i}z)$. Their **Wronskian** determines

$$g = \frac{w^2(w^2-1)(w^2-9)f^3}{2^6 3^4} = \frac{\eta_2^5\eta_3^4\eta_6}{\eta_1^4} = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3} = \sum_{n=1}^\infty \frac{n^2(q^n - q^{5n})}{1-q^{6n}}.$$

With $\chi_6(n) := \chi_2(n)\chi_3(n)$ and $\chi_2(n) := (1-(-1)^n)/2$, the **solution**

$$\frac{I(w^2)}{f} = \frac{\pi\log(-1/q)}{\sqrt{3}} - 3\sum_{n=1}^\infty \frac{\chi_6(n)}{n^2}\frac{1+q^n}{1-q^n}$$

is obtained by making $I(1)$ finite, as was shown by **Bloch** and **Vanhove**.

1.1 Chan-Zudilin transformations of the BFT equation

$$z_2 := \frac{2z-1}{6z-2}, \quad z_3 := \frac{3z-2}{6z-3}, \quad z_6 := \frac{-1}{6z}, \quad q_k := \exp(2\pi \mathrm{i} z_k),$$
$$f_2(z) := \frac{\eta_2^6\eta_3}{\eta_1^3\eta_6^2}, \quad f_3(z) := \frac{\eta_2\eta_3^6}{\eta_1^2\eta_6^3}, \quad f_6(z) := \frac{\eta_1\eta_6^6}{\eta_2^2\eta_3^3},$$
$$-\left(q_k\frac{\mathrm{d}}{\mathrm{d}q_k}\right)^2 \frac{I(w^2)}{6f_k(z_k)} = g_k(z_k),$$
$$g_2(z) := \frac{\eta_1^5\eta_3\eta_6^4}{\eta_2^4}, \quad g_3(z) := \frac{\eta_1^4\eta_2\eta_6^5}{\eta_3^4}, \quad g_6(z) := \frac{\eta_1\eta_2^4\eta_3^5}{\eta_6^4},$$

from which I obtain **alternative expansions**

$$\begin{aligned}
\frac{I(w^2)}{f_2(z_2)} &= \sum_{n=1}^{\infty}\frac{3\chi_3(n)}{n^2}\frac{(1-q_2^n)^2}{1+q_2^{2n}} = I(0) - \sum_{n=1}^{\infty}\frac{6\chi_3(n)}{n^2}\frac{q_2^n}{1+q_2^{2n}},\\
\frac{I(w^2)}{f_3(z_3)} &= \sum_{n=1}^{\infty}\frac{2\chi_2(n)}{n^2}\frac{(1-q_3^n)^3}{1-q_3^{3n}} = I(1) - \sum_{n=1}^{\infty}\frac{6\chi_2(n)}{n^2}\frac{q_3^n}{1+q_3^n+q_3^{2n}},\\
\frac{I(w^2)}{f_6(z_6)} &= -3\log^2(-q_6) + \sum_{n=1}^{\infty}\frac{6}{n^2}\frac{q_6^n}{1-q_6^n+q_6^{2n}}.
\end{aligned}$$

1.2 Optimal choice of nome

Use a **real** nome $q = Q(w)$ for $w > 1$, or $q_k = Q(w_k)$ for $w_k > 1$, with

$$Q(w) := \exp\left(\frac{-\pi\mathbf{agm}(1,\sqrt{r})}{\mathbf{agm}(1,\sqrt{1-r})}\right), \quad r = \frac{16w}{(w+3)(w-1)^3},$$
$$w_2^2 = \frac{w^2-9}{w^2-1}, \quad w_3^2 = \frac{9}{w^2}, \quad w_6^2 = \frac{9}{w_2^2}.$$

For real w^2 there are **two** choices. Use the nome with **faster** convergence.

1.3 Eta quotients chosen by Adams, Bogner and Weinzierl

The **Mainz** group use $q_M := -q_2$, thereby encountering $\eta_{\mathbf{4}}$ and $\eta_{\mathbf{12}}$ in

$$f_M(z) := f_2\left(z+\frac{1}{2}\right) = \frac{\eta_1^3\eta_4^3\eta_6}{\eta_2^3\eta_3\eta_{12}}, \quad g_M(z) := g_2\left(z+\frac{1}{2}\right) = -\frac{\eta_2^{11}\eta_6^7}{\eta_1^5\eta_3\eta_4^5\eta_{12}}.$$

1.4 Eta quotients at DESY, Linz and Tallahassee

Ablinger, Blümlein, De Freitas, van Hoeij, Imamoglu, Raab, Radu and **Schneider** encountered a second-order equation with complicated coefficients and powers of $\log(x)$ in the inhomogeneous term. Their **homogeneous** equation has a **hypergeometric** solution

$$H(x) = \frac{(x^2-1)^2}{9(x^2+3)} \sum_{n=0}^{\infty} \frac{(\mathbf{4/3})_n(\mathbf{5/3})_n}{n!(n+1)!} \left(\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right)^{n+1} = \frac{x^2}{9} + O(x^4)$$

with **Pochhammer** parameters, mod 1, among the 8 **triangle** group cases

$$\left(\frac{1}{12}, \frac{5}{12}\right), \left(\frac{1}{8}, \frac{3}{8}\right), \left(\frac{1}{6}, \frac{1}{3}\right), \left(\frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{3}, \frac{1}{2}\right), \left(\mathbf{\frac{1}{3}}, \mathbf{\frac{2}{3}}\right),$$

that are transforms of the **elliptic** case $(\frac{1}{2}, \frac{1}{2})$. Thus I obtained

$$\begin{aligned} H\left(3\frac{\eta_1^2\eta_6^4}{\eta_2^4\eta_3^2}\right) &= \frac{1}{2}\left(\frac{\eta_1^{14}\eta_6^{10}}{\eta_2^{22}\eta_3^2} + \frac{\eta_1^6\eta_6^4}{\eta_2^{12}\eta_3^2}\left(\frac{\eta_1^4\eta_6^8}{\eta_2^8\eta_3^4} + \frac{1}{3}\right) q\frac{\mathrm{d}}{\mathrm{d}q}\right)\frac{\eta_2^6\eta_3}{\eta_1^3\eta_6^2} \\ &= q - 6q^2 + 24q^3 - 74q^4 + 195q^5 - 474q^6 + 1100q^7 + O(q^8) \end{aligned}$$

as a **modular parametrization**, verified by expanding up to q^{1000}.

2 Three-loop sunrise

Bailey, Borwein, Broadhurst and **Glasser** developed the expansion of

$$J(t) = 8\int_0^\infty I_0(\sqrt{t}x)K_0^4(x)x\mathrm{d}x = 7\zeta(3) + O(t)$$

about $t = 0$. A neat q-expansion comes from a transformation by **Joyce**:

$$t := 10 - w^2 - \frac{9}{w^2} = -64\left(\frac{\eta_2\eta_6}{\eta_1\eta_3}\right)^6 = -64q + O(q^2)$$

$$\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^3 \frac{J(t)}{(wf/3)^2} = 24h = 24\sum_{n=1}^\infty \frac{n^3(q^n - 8q^{3n} + q^{5n})}{1 - q^{6n}}$$

$$\frac{J(t)}{(wf/3)^2} = 12\sum_{n=1}^\infty \frac{\phi(n)}{n^3}\frac{1+q^n}{1-q^n} = J(0) + 24\sum_{n=1}^\infty \frac{\phi(n)}{n^3}\frac{q^n}{1-q^n}$$

with $\phi(n) = 0, \mathbf{1}, 0, -\mathbf{8}, 0, \mathbf{1}$, for $n = 0, 1, 2, 3, 4, 5 \bmod 6$. This **novel** expansion works well for $t \in [-8, 8]$, where $|q| \le \exp(-\sqrt{2}\pi/3) < 0.22742$.

2.1 Chan-Zudilin transformation at three loops

Here I use the Fourier expansion of

$$h_6(z) := \frac{-64h}{t} = 1 + 2h - 30\sum_{n=1}^{\infty}\frac{n^3(q^{2n}+q^{4n}-8q^{6n})}{1-q^{6n}}.$$

Transforming to $q_6 := \exp(2\pi \mathrm{i} z_6)$, with $z_6 := \frac{-1}{6z}$, I obtain

$$\left(q_6\frac{\mathrm{d}}{\mathrm{d}q_6}\right)^3 \frac{J(t)}{(wf_6(z_6))^2} = -24h_6(z_6)$$

$$\frac{J(t)}{(wf_6(z_6))^2} = -4\log^3(q_6) + 24\sum_{n=1}^{\infty}\frac{15\phi(n+3)-\phi(n)}{n^3}\frac{1+q_6^n}{1-q_6^n}.$$

Bloch, Kerr and **Vanhove** obtained this for $t \le 4$, **neglecting** issues of analytic continuation or efficient convergence, which I shall now **resolve**.

2.2 Determination of nomes

For all finite real t, the following procedure in `Pari-GP`

```
Z(t)={x=2/(sqrt(4-t)+sqrt(16-t));a=sqrt((1-x)^3*(1+3*x));
I/2*agm(a,4*x*sqrt(x))/agm(a,sqrt((1+x)^3*(1-3*x)))}
```

determines z and hence the nomes q and q_6. Then the expansions in q and q_6 **agree** for $t \le 16$. For $t > 16$, they differ by **complex conjugation**. Expansion in q_6 gives a **positive** imaginary part on the top lip of the cut.

2.3 Dispersion relation

$$J(u^2) = \int_{16}^{\infty} \frac{f_-^2(t) - 3f_+^2(t)}{t-u^2} \mathrm{d}t, \quad f_\pm(t) = \frac{2\pi}{\mathbf{agm}\left(\sqrt{2\mu}, \sqrt{\mu \pm \sqrt{\nu}}\right)},$$

$$\mu = \frac{2\alpha+\beta}{2}, \quad \nu = \mu^2 - 48\frac{\alpha-\beta}{\alpha+\beta}, \quad \alpha = \sqrt{t^2-4t}, \quad \beta = \sqrt{t^2-16t}.$$

2.4 Evaluation at the fifteenth singular value

As **Laporta** and I discovered at Bielefeld in 2007, the **on-shell** integral

$$J(1)=\frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30\sqrt{5}}$$

has a stunningly simple evaluation. Its **proof** by Bloch, Kerr and Vanhove, elucidated by Samart, was **complicated** by expansion in complex q_6 and by a character $\psi(n):=720\phi(n+3)-15\phi(n)$ with support at even integers. By **contrast**, expansion in q reduced the burden of proof to showing that

$$\sum_{n=1}^{\infty}\frac{\phi(n)}{n^3}\frac{1-\exp(-\pi\sqrt{5/3}n)}{1+\exp(-\pi\sqrt{5/3}n)}=\frac{\pi^3}{12\sqrt{15}}$$

which results from an **Eichler integral** that is **rational**:

$$\int_0^{\sqrt{1/15}}h\left(\frac{1+iy}{2}\right)y\mathrm{d}y=\frac{1}{120},\quad h(z)=\frac{\eta_2^{16}}{\eta_1^8}-9\frac{\eta_6^{16}}{\eta_3^8}.$$

3 Eichler integrals, on shell, up to six loops

3.1 A notable Eichler integral at two loops

The two-loop **dispersion relation** evaluates an Eichler integral **on shell**:

$$I(v^2) = \frac{2\pi}{\sqrt{3}} \int_3^\infty \frac{wf\mathrm{d}w}{w^2 - v^2}, \quad I(1) = \frac{\pi^2}{4},$$

$$\frac{wf}{w^2-1}\frac{\mathrm{d}w}{\mathrm{d}z} = 9\pi \mathrm{i} g_2, \quad g_2(z) := \frac{\eta_1^5 \eta_3 \eta_6^4}{\eta_2^4} = \frac{\eta_3^9}{\eta_1^3} - 8\frac{\eta_6^9}{\eta_2^3},$$

$$\int_0^\infty g_2(\mathrm{i}y)\mathrm{d}y = \frac{1}{\sqrt{\mathbf{1728}}}.$$

3.2 Eichler integrals at three loops

Ten years ago I discovered that

$$C := \frac{\pi}{16}\left(1 - \frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty} \exp\left(-\pi\sqrt{15}n^2\right)\right)^4 = \frac{1}{240\sqrt{5}\pi^2}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{15}\right)$$

and its **reciprocal** determine four **Bessel moments** of the form

$$M(a,b,c) := \int_0^\infty I_0^a(x) K_0^b(x) x^c \mathrm{d}x.$$

3.3 Evaluation of a matrix of Bessel moments

$$\mathcal{M}_5 := \left[\begin{array}{cc} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{array}\right] = \left[\begin{array}{cc} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{array}\right].$$

A **proof** of my old conjecture was given by **Yajun Zhou** who expanded

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^3} \frac{q^n}{1-q^n} = 4\pi^3 \mathrm{i} \int_z^{\mathrm{i}\infty} h(z')(z'-z)^2 \mathrm{d}z', \quad h(z) = \frac{\eta_2^{16}}{\eta_1^8} - 9\frac{\eta_6^{16}}{\eta_3^8},$$

about $z = \frac{1+\mathrm{i}\sqrt{5/3}}{2}$. Evaluation of $M(1,4,3)$ requires the Eichler integrals

$$\begin{aligned} -\int_{\sqrt{5/3}}^{\infty} h\left(\frac{1+\mathrm{i}y}{2}\right) \mathrm{d}y &= \frac{\sqrt{3}I(0)}{80\pi^2}, \\ -\int_{\sqrt{5/3}}^{\infty} h\left(\frac{1+\mathrm{i}y}{2}\right) y\mathrm{d}y &= \frac{1}{\mathbf{120}} = \int_0^{\sqrt{1/15}} h\left(\frac{1+\mathrm{i}y}{2}\right) y\mathrm{d}y, \\ -\int_{\sqrt{5/3}}^{\infty} h\left(\frac{1+\mathrm{i}y}{2}\right) y^2\mathrm{d}y &= \frac{J(0)}{12\pi^3} - \frac{I(0)}{16\sqrt{3}\pi^2}. \end{aligned}$$

It is notable that the **deteminant** $\det \mathcal{M}_5 = 2\pi^3/\sqrt{3^3 5^5}$ is **free** of C.

3.4 Critical L-series for moments of 5 Bessel functions

Consider the **Fourier** expansion of the weight-3 level-15 **cuspform**

$$f_{3,15}(z) := (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n) q^n = -\frac{f_{3,15}(-1/(15z))}{(-15)^{3/2}z^3}.$$

If the **Kronecker** symbol $\left(\frac{p}{15}\right) = \left(\frac{p}{3}\right)\left(\frac{p}{5}\right)$ is negative, for **prime** p, then $A_5(p) = 0$. For $\Re s > 2$, there is a convergent **L-series**

$$L_5(s) = \sum_{n>0} \frac{A_5(n)}{n^s} = \prod_p \frac{1}{1 - A_5(p)p^{-s} + \left(\frac{p}{15}\right)p^{2-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_5(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{3,15}(\mathrm{i}y) y^{s-1} \mathrm{d}y$$

with **critical** values, from a result by **Rogers, Wan** and **Zucker**,

$$L_5(\mathbf{1}) = \frac{5}{\pi^2} \int_0^\infty I_0(x) K_0^4(x) x \mathrm{d}x, \quad L_5(\mathbf{2}) = \frac{4}{3} \int_0^\infty I_0^2(x) K_0^3(x) x \mathrm{d}x.$$

3.5 Critical L-series for moments of 6 Bessel functions

Consider the **Fourier** expansion of the weight-4 level-6 **cuspform**

$$f_{4,6}(z) := (\eta_1\eta_2\eta_3\eta_6)^2 = \sum_{n>0} A_6(n)q^n = \frac{f_{4,6}(-1/(6z))}{6^2 z^4}.$$

For $\Re s > 5/2$, there is a convergent **L-series**

$$L_6(s) = \sum_{n>0} \frac{A_6(n)}{n^s} = \frac{1}{1+2^{1-s}} \frac{1}{1+3^{1-s}} \prod_{p>3} \frac{1}{1 - A_6(p)p^{-s} + p^{3-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_6(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{4,6}(\mathrm{i}y) y^{s-1} \mathrm{d}y$$

with **critical** values related to **Bessel moments** as follows

$$L_6(\mathbf{2}) = \frac{2}{\pi^2} M(1,5,1) = \frac{2}{3} M(3,3,1),$$
$$L_6(\mathbf{1}) = \frac{2}{\pi^2} M(2,4,1) = \frac{3}{\pi^2} L_6(\mathbf{3}).$$

3.6 Critical L-series for a moment of 7 Bessel functions

With 7 Bessel functions and $\Re s > 3$, the local factors at the primes in

$$L_7(s) = \prod_p \frac{1}{Z_7(p, p^{-s})}$$

are given, for p coprime to 105, by the **cubic**

$$Z_7(p,T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |\lambda_p|^2)T + p^4 T^2\right)$$

where λ_p is a complex Hecke eigenvalue of a weight-3 newform with level 525. For $p|105$, I obtained, from **Kloosterman** moments in **finite fields**,

$$Z_7(3,T) = 1 - 10T + 3^4 T^2,\ Z_7(5,T) = 1 - 5^4 T^2,\ Z_7(7,T) = 1 + 70T + 7^4 T^2.$$

Then **Anton Mellit** suggested a **functional equation**

$$\Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_7(s) = \Lambda_7(5-s)$$

and **Tim Dokchitser**'s package ComputeL gave the empirical result

$$L_7(\mathbf{2}) \stackrel{?}{=} \frac{24}{5\pi^2} \int_0^\infty I_0^2(x) K_0^5(x) x \mathrm{d}x.$$

3.7 Critical L-series for moments of 8 Bessel functions

Consider the **Fourier** expansion of the weight-6 level-6 **cuspform**

$$f_{6,6}(z) := \frac{\eta_2^9\eta_3^9}{\eta_1^3\eta_6^3} + \frac{\eta_1^9\eta_6^9}{\eta_2^3\eta_3^3} = \sum_{n>0} A_8(n)q^n = -\frac{f_{6,6}(-1/(6z))}{6^3 z^6}.$$

For $\Re s > 7/2$, there is a convergent **L-series**

$$L_8(s) = \sum_{n>0} \frac{A_8(n)}{n^s} = \frac{1}{1-2^{2-s}}\frac{1}{1+3^{2-s}}\prod_{p>3}\frac{1}{1-A_8(p)p^{-s}+p^{5-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_8(s) = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f_{6,6}(\mathrm{i}y)y^{s-1}\mathrm{d}y$$

with **critical** values related to **Bessel moments** as follows

$$L_8(\mathbf{4}) = \frac{4}{9\pi^2}M(1,7,1) = \frac{4}{9}M(3,5,1) = \frac{\pi^2}{9}L_8(\mathbf{2}),$$

$$L_8(\mathbf{5}) = \frac{4}{27}M(2,6,1) = \frac{2\pi^2}{21}M(4,4,1) = \frac{2\pi^2}{21}L_8(\mathbf{3}) = \frac{\pi^4}{54}L_8(\mathbf{1}).$$

3.8 Conjectures for non-critical L-series

Anton Mellit and I discovered and **no-one** has yet proved that

$$\det \int_0^\infty K_0^3(x) \begin{bmatrix} K_0^2(x) & x^2 K_0^2(x) \\ I_0^2(x) & x^2 I_0^2(x) \end{bmatrix} x\mathrm{d}x \stackrel{?}{=} \frac{45}{8\pi^2} L_5(4)$$
$$\det \int_0^\infty K_0^4(x) \begin{bmatrix} K_0^2(x) & x^2 K_0^2(x) \\ I_0^2(x) & x^2 I_0^2(x) \end{bmatrix} x\mathrm{d}x \stackrel{?}{=} \frac{27}{4\pi^2} L_6(5)$$
$$\det \int_0^\infty K_0^6(x) \begin{bmatrix} K_0^2(x) & x^2(1-2x^2) K_0^2(x) \\ I_0^2(x) & x^2(1-2x^2) I_0^2(x) \end{bmatrix} x\mathrm{d}x \stackrel{?}{=} \frac{6075}{128\pi^2} L_8(7).$$

Recently, **Yajun Zhou** showed that the first two of these conjectures are equivalent to conjectures on logarithmic **Mahler** measures by **Fernando Rodriguez Villegas**. The third appears to have no such counterpart.

4 Candidates for quasi-periods

Laporta encountered the **moments** in the first row of the **determinant**

$$\det\begin{bmatrix} M(1,5,1) & 4M(1,5,3) \\ M(2,4,1) & 4M(2,4,3) \end{bmatrix} = \frac{\pi^4}{144}$$

at 4 loops in **quantum electrodynamics**. In terms of **Eichler** integrals,

$$\begin{aligned}
\frac{D_2}{2} &= \frac{M(1,5,1)}{\pi^4} = \frac{4M(1,5,3)}{\pi^4} + \frac{5E_2}{18} \\
\frac{3D_1}{5} &= \frac{M(2,4,1)}{\pi^3} = \frac{4M(2,4,3)}{\pi^3} + \frac{E_1}{3} \\
\begin{bmatrix} D_s \\ E_s \end{bmatrix} &:= -\int_{1/\sqrt{3}}^{\infty} \begin{bmatrix} f_{6,4}\left(\frac{1+\mathrm{i}y}{2}\right) \\ g_{6,4}\left(\frac{1+\mathrm{i}y}{2}\right) \end{bmatrix} y^{s-1}\mathrm{d}y, \\
g_{4,6}(z) := \frac{(w^2-3)^2(w^4+9)}{8w^4} & f_{4,6}(z) = 5q + 102q^2 + 945q^3 + O(q^4), \\
D_1E_2 - D_2E_1 &= \frac{1}{24\pi^3}.
\end{aligned}$$

A link to **Francis Brown**'s concept of **quasi-periods** emerged from discussion this month, in Bonn.

4.1 Quasi-periods at six loops?

I have **empirical** relations to Eichler integrals for the **second** column of

$$\det\left[\begin{array}{cc} M(1,7,1) & 32M(1,7,3)-64M(1,7,5) \\ M(2,6,1) & 32M(2,6,3)-64M(2,6,5)\end{array}\right] = \frac{5\pi^6}{192},$$

$$\frac{F_2}{4} = \frac{M(1,7,1)}{\pi^6} \overset{?}{=} \frac{32M(1,7,3)-64M(1,7,5)}{\pi^6} + \frac{35G_2}{108},$$

$$\frac{9F_1}{28} = \frac{M(2,6,1)}{\pi^5} \overset{?}{=} \frac{32M(2,6,3)-64M(2,6,5)}{\pi^5} + \frac{5G_1}{12},$$

$$\left[\begin{array}{c} F_s \\ G_s \end{array}\right] := -\int_{1/\sqrt{3}}^{\infty} \left[\begin{array}{c} f_{6,6}\left(\frac{1+\mathrm{i}y}{2}\right) \\ g_{6,6}\left(\frac{1+\mathrm{i}y}{2}\right) \end{array}\right] (3y^2-1)y^{s-1}\mathrm{d}y,$$

$$g_{6,6}(z) := \frac{(w^2-3)^4}{16w^4} f_{6,6}(z) = q + 36q^2 + 567q^3 + 5264q^4 + O(q^5),$$

$$F_1G_2 - F_2G_1 \overset{?}{=} \frac{1}{4\pi^5},$$

with $(3y^2-1)$ inferred from the **dispersion relation** for a sub-diagram. Note that the integrand of G_s is of order $(3y^2-1)^6$ near its threshold. A link to **Francis Brown**'s concept of **quasi-periods** is forming, yet is not complete, since $g_{6,6}$ lacks a period polynomial enjoyed by $f_{6,6}$.

4.2 Six-loop sunrise for algebraic geometers

Sunrise integrals are also given by integrals over **Schwinger** parameters:

$$S_{L,k} := \int_0^\infty \frac{\mathrm{d}x_1}{x_1} \dots \int_0^\infty \frac{\mathrm{d}x_L}{x_L} \left(\left(1 + \sum_{i=1}^{L} x_i\right) \left(1 + \sum_{j=1}^{L} \frac{1}{x_j}\right) - 1 \right)^{-k},$$

$$2^L M(1, L+1, 1) = S_{L,1}, \quad 2^{L-2} M(1, L+1, 3) = S_{L,2} + 2S_{L,3},$$
$$2^{L-6} M(1, L+1, 5) = S_{L,3} + 6S_{L,4} + 6S_{L,5}.$$

Conjecturally, G_2 is a **Q**-linear combination of $S_{6,k}/\pi^6$, for $k = 1$ to 5. Thanks to **eta quotients**, it is easily evaluated at high precision:

$$g_{6,6}(z) = \left(\frac{3\eta_2^2\eta_3^4}{2\eta_1^4\eta_6^2} - \frac{\eta_1^4\eta_6^2}{2\eta_2^2\eta_3^4} \right)^4 \left(\frac{\eta_2^9\eta_3^9}{\eta_1^3\eta_6^3} + \frac{\eta_1^9\eta_6^9}{\eta_2^3\eta_3^3} \right) = \sum_{n>0} g_6(n) q^n,$$

$$G_2 = 18 \sum_{n>0} \frac{(-1)^{n-1} g_6(n)}{(n\pi)^4} \left(1 + \frac{n\pi}{\sqrt{3}} + \frac{n^2\pi^2}{9}\right) \exp\left(\frac{-n\pi}{\sqrt{3}}\right).$$

4.3 Quasi-periods on $\Gamma_0(8)$

$$
\left[\begin{array}{cc} 2M(0,4,0) & 4M(0,4,0)-16M(0,4,2) \\ 2M(1,3,0) & 4M(1,3,0)-16M(1,3,2) \end{array}\right] = \left[\begin{array}{cc} \pi^4 P_1 & 3\pi^4 Q_1 \\ \pi^3 P_2 & 3\pi^3 Q_2 \end{array}\right],
$$

$$
\left[\begin{array}{c} P_s \\ Q_s \end{array}\right] := -\mathrm{i}\int_1^\infty \left[\begin{array}{c} f_{4,8}\left(\frac{1+\mathrm{i}y}{4}\right) \\ g_{4,8}\left(\frac{1+\mathrm{i}y}{4}\right) \end{array}\right] \frac{y^s+y^{4-s}}{y}\mathrm{d}y,
$$

$$
f_{4,8}(z) := (\eta_2\eta_4)^4 = q - 4q^3 - 2q^5 + O(q^7),
$$

$$
g_{4,8}(z) := \left(1+64\frac{\eta_4^{24}}{\eta_2^{24}}\right) f_{4,8}(z) = q + 60q^3 + 1278q^5 + O(q^7),
$$

$$
P_1Q_2 - P_2Q_1 = -\frac{1}{2\pi^3},
$$

with $g_{4,8}(z_0) = 0$ at $z_0 = (1+\mathrm{i})/4$, where $-\mathrm{i}f_{4,8}(z_0) = \Gamma^8(1/4)/(128\pi^6)$.

4.4 Periods on $\Gamma_0(24)$

The **unique** weight-6 Hecke eigenform that is both a **newform** on $\Gamma_0(24)$ and also has a **negative** sign in the functional equation for its **L-series** is

$$\begin{aligned} f_{6,24}(z) &:= \frac{\eta_3^4\eta_4^2\eta_6^6\eta_8^2}{\eta_{24}^2} + \frac{\eta_1^4\eta_2^6\eta_{12}^2\eta_{24}^2}{3\eta_8^2} - \frac{16\eta_1^2\eta_2^2\eta_{12}^6\eta_{24}^4}{\eta_3^2} - \frac{16\eta_3^2\eta_4^6\eta_6^2\eta_8^4}{3\eta_1^2} \\ &\quad + \frac{64\eta_1^2\eta_3^2\eta_4\eta_8^4\eta_{12}\eta_{24}^4}{\eta_2\eta_6} - \frac{4\eta_1^4\eta_2\eta_3^4\eta_6\eta_8^2\eta_{24}^2}{\eta_4\eta_{12}} = -f_{6,24}(z+1/2) \\ &= \frac{f_{6,24}(-1/(24z))}{24^3 z^6} = -\frac{f_{6,24}((3z-1)/(12z-3))}{3^3(4z-1)^6} \\ &= q - 9q^3 - 34q^5 - 240q^7 + 81q^9 - 124q^{11} + 46q^{13} + O(q^{15}). \end{aligned}$$

David Roberts and I related its **critical** L-series to **Bessel** moments:

$$\begin{aligned} \widetilde{L}_6(4) &\stackrel{?}{=} \frac{M(0,6,0)}{108\pi^2}, \\ \widetilde{L}_6(5) &\stackrel{?}{=} \frac{M(1,5,0)}{144}, \\ \widetilde{L}_6(s) &:= \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{6,24}(\mathrm{i}y) y^{s-1} \mathrm{d}y. \end{aligned}$$

4.5 Striving for quasi-periods on $\Gamma_0(24)$

After **intensive** experiment at high precision, I **conjecture** that

$$\det\begin{bmatrix} M(0,6,0) & 3M(0,6,2)-8M(0,6,4) \\ M(1,5,0) & 3M(1,5,2)-8M(1,5,4) \end{bmatrix} \stackrel{?}{=} \frac{5\pi^6}{16},$$

$$\frac{M(0,6,0)}{\pi^6} \stackrel{?}{=} \frac{R_1}{28} \stackrel{?}{=} \frac{3R_3}{4}, \qquad \frac{M(1,5,0)}{\pi^5} \stackrel{?}{=} \frac{R_2}{8},$$

$$R_s := -\mathrm{i}\int_0^\infty f_{6,24}\left(\frac{1+\mathrm{i}y}{4}\right) y^{s-1}\mathrm{d}y = 3^{3-s}R_{6-s}.$$

I strove to relate the **second** column of the determinant to Eichler integrals of another modular function. Perhaps **Francis Brown** can accomplish this. Perhaps **Yajun Zhou** can elucidate the **first** column.

5 Critical L-series up to 22 loops

Let $\Omega_{a,b}$ be the **determinant** of the $r \times r$ matrix with $M(a,b,1)$ at top left, size $r = \lceil (a+b)/4 - 1 \rceil$, powers of x^2 increasing to the right and powers of $I_0^2(x)$ increasing downwards. Thus $\Omega_{1,23}$ is a 5×5 determinant with the **22-loop sunrise** integral $M(1,23,1)$ at **top left** and $M(9,15,9)$ at bottom right. **David Roberts** and I discovered that

$$\begin{aligned}
L_8(4) &= \frac{2^2\,\Omega_{1,7}}{3^2\pi^2} \equiv \frac{4}{9\pi^2}\int_0^\infty I_0(t)K_0^7(t)t\mathrm{d}t \\
L_{12}(6) &\stackrel{?}{=} \frac{2^6\,\Omega_{1,11}}{3^4 \times 5\pi^6} \\
L_{16}(8) &\stackrel{?}{=} \frac{2^{14}\,\Omega_{1,15}}{3^7 \times 5^2 \times 7\pi^{12}} \\
L_{20}(10) &\stackrel{?}{=} \frac{2^{22} \times 11 \times \mathbf{131}\,\Omega_{1,19}}{3^{11} \times 5^6 \times 7^3\pi^{20}} \qquad \text{to 44 digits} \\
L_{24}(12) &\stackrel{?}{=} \frac{2^{29} \times \mathbf{12558877}\,\Omega_{1,23}}{3^{19} \times 5^9 \times 7^3 \times 11\pi^{30}} \qquad \text{to 19 digits,}
\end{aligned}$$

where boldface highlights **primes** $p > N = a + b$. **30 GHz-years** of work gave 44-digit **precision** for $L_{20}(10)$. $L_{24}(12)$ agrees up to 19 digits.

With a **cut** of a line in the diagram at top left of the matrix, we found

$$
\begin{aligned}
L_8(5) &= \frac{2^2\,\Omega_{2,6}}{3^3} \equiv \frac{4}{27}\int_0^\infty I_0^2(t)K_0^6(t)t\mathrm{d}t\\
L_{12}(7) &\stackrel{?}{=} \frac{2^5\times 11\,\Omega_{2,10}}{3^6\times 5^2\pi^2}\\
L_{16}(9) &\stackrel{?}{=} \frac{2^{14}\times 13\,\Omega_{2,14}}{3^9\times 5^3\times 7^2\pi^6}\\
L_{20}(11) &\stackrel{?}{=} \frac{2^{19}\times 17\times 19\times \mathbf{23}\,\Omega_{2,18}}{3^{13}\times 5^7\times 7^3\pi^{12}}\\
L_{24}(13) &\stackrel{?}{=} \frac{2^{27}\times 17\times 19^2\times 23^2\times \mathbf{46681}\,\Omega_{2,22}}{3^{23}\times 5^{12}\times 7^4\times 11^2\pi^{20}}\,.
\end{aligned}
$$

Enlarged determinants $\widehat{\Omega}_{2,4r+2}$ of size $r+1$, with **regularization** of $M(2r+2,2r+2,2r+1)$ at bottom right, determine the central derivatives

$$
\begin{aligned}
-L'_{12}(5) &\stackrel{?}{=} \frac{2^4\left(2^6\times \mathbf{29}\,\widehat{\Omega}_{2,10}+3\,\Omega_{2,10}\log 2\right)}{3^2\times 7\pi^6}\\
-L'_{16}(7) &\stackrel{?}{=} \frac{2^9\left(2^7\times \mathbf{83}\,\widehat{\Omega}_{2,14}+3\times 11\,\Omega_{2,14}\log 2\right)}{3^5\times 5\pi^{12}}\\
-L'_{20}(9) &\stackrel{?}{=} \frac{2^{17}\times 17\times 19\left(2^9\times 7\times \mathbf{101}\,\widehat{\Omega}_{2,18}+5\times 13\,\Omega_{2,18}\log 2\right)}{3^8\times 5^4\times 7^2\times 11\pi^{20}}\,.
\end{aligned}
$$

In the cases with $N = 4r + 2$, we obtained

$$\begin{aligned}
L_6(2) &= \frac{2\,\Omega_{1,5}}{\pi^2} \equiv \frac{2}{\pi^2}\int_0^\infty I_0(t)K_0^5(t)t\mathrm{d}t \\
L_6(3) &= \frac{2\,\Omega_{2,4}}{3} \equiv \frac{2}{3}\int_0^\infty I_0^2(t)K_0^4(t)t\mathrm{d}t \\
L_{10}(4) &\stackrel{?}{=} \frac{2^7\,\Omega_{1,9}}{3^2\pi^6} \\
L_{10}(5) &\stackrel{?}{=} \frac{2^4\,\Omega_{2,8}}{3\times 5\pi^2} \\
L_{14}(6) &= 0 \\
L_{14}(7) &\stackrel{?}{=} \frac{2^{10}\times 11\times 13\,\Omega_{2,12}}{3^6\times 5^2\times 7\pi^6} \\
L_{18}(8) &\stackrel{?}{=} \frac{2^{21}\times 17\times \mathbf{19}\,\Omega_{1,17}}{3^5\times 5^4\times 7\pi^{20}} \\
L_{18}(9) &\stackrel{?}{=} \frac{2^{12}\times 13\times 17\times \mathbf{41}\,\Omega_{2,16}}{3^8\times 5^3\times 7^2\pi^{12}} \\
L_{22}(10) &= 0 \\
L_{22}(11) &\stackrel{?}{=} \frac{2^{23}\times 17\times 19\times \mathbf{11621}\,\Omega_{2,20}}{3^{14}\times 5^7\times 7^3\pi^{20}}
\end{aligned}$$

with central vanishing from an odd sign at $N = 14$ and $N = 22$.

For cases with odd N, we obtained

$$\begin{aligned}
L_5(2) &= \frac{2^2\,\Omega_{2,3}}{3} \equiv \frac{4}{3}\int_0^\infty I_0^2(t)K_0^3(t)t\mathrm{d}t \\
L_7(2) &\stackrel{?}{=} \frac{2^3\times 3\,\Omega_{2,5}}{5\pi^2} \equiv \frac{24}{5\pi^2}\int_0^\infty I_0^2(t)K_0^5(t)t\mathrm{d}t \\
L_9(4) &\stackrel{?}{=} \frac{2^6\,\Omega_{2,7}}{3\times 5\pi^2} \\
L_{11}(4) &\stackrel{?}{=} \frac{2^8\times 5\,\Omega_{2,9}}{3\times 7\pi^6} \\
L_{13}(6) &\stackrel{?}{=} \frac{2^7\times \mathbf{149}\,\Omega_{2,11}}{3^3\times 5\times 7\pi^6} \\
L_{15}(6) &\stackrel{?}{=} \frac{2^8\times 7\times \mathbf{53}\,\Omega_{2,13}}{3^2\times 5\pi^{12}} && \text{to 43 digits} \\
L_{17}(8) &\stackrel{?}{=} \frac{2^{15}\times \mathbf{29}\,\Omega_{2,15}}{3^5\times 5^2\times 7\pi^{12}} && \text{to 23 digits} \\
L_{19}(8) &\stackrel{??}{=} \frac{2^{14}\times \mathbf{1093}\times \mathbf{13171}\,\Omega_{2,17}}{3^4\times 5^4\times 7\times 11\pi^{20}} && \text{to 14 digits.}
\end{aligned}$$

Comment: We also have results relating Bessel moments $M(a,b,c)$ with **even** c to L-series from Kloosterman moments with a quadratic **twist**.

6 Quadratic relations between periods, for all loops

Conjecture: *With the Betti, period and de Rham matrices below,*

$$B_N = P_N D_N P_N^{\mathtt{tr}}.$$

Construction: The **Betti** matrices have **rational** elements given by

$$\begin{aligned} B_{2k+1}(u,v) &= (-1)^{u+k}2^{-2k-2}(k+u)!(k+v)!Z(u+v) \\ B_{2k+2}(u,v) &= (-1)^{u+k}2^{-2k-3}(k+u+1)!(k+v+1)!Z(u+v+1) \\ Z(m) &= \frac{1+(-1)^m}{(2\pi)^m}\zeta(m) \end{aligned}$$

with u and v running from 1 to k.

The elements of the **period** matrices are given by Bessel moments

$$\begin{aligned} P_{2k+1}(u,a) &\equiv \frac{(-1)^{a-1}}{\pi^u} M(k+1-u, k+u, 2a-1) \\ P_{2k+2}(u,a) &\equiv \frac{(-1)^{a-1}}{\pi^{u+1/2}} M(k+1-u, k+1+u, 2a-1). \end{aligned}$$

Our construction of the rational **de Rham** matrices was highly inductive.

Let v_k and w_k be the rational numbers **generated** by

$$\begin{aligned}\frac{J_0^2(t)}{C(t)} &= \sum_{k\geq 0}\frac{v_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1-\frac{17t^2}{54}+\frac{3781t^4}{186624}+\ldots\\ \frac{2J_0(t)J_1(t)}{tC(t)} &= \sum_{k\geq 0}\frac{w_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1-\frac{41t^2}{216}+\frac{325t^4}{186624}+\ldots\end{aligned}$$

where $J_0(t) = I_0(\mathrm{i}t)$, $J_1(t) = -J_0'(t)$ and

$$C(t) \equiv \frac{32(1-J_0^2(t)-tJ_0(t)J_1(t))}{3t^4} = 1-\frac{5t^2}{27}+\frac{35t^4}{2304}-\frac{7t^6}{9600}+\ldots$$

We construct rational bivariate polynomials by the **recursion**

$$\begin{aligned}H_s(y,z) &= zH_{s-1}(y,z)-(s-1)yH_{s-2}(y,z)\\ &- \sum_{k=1}^{s-1}\binom{s-1}{k}\left(v_kH_{s-k}(y,z)-w_kzH_{s-k-1}(y,z)\right)\end{aligned}$$

for $s>0$, with $H_0(y,z)=1$. We use these to define

$$d_s(N,c) \equiv \frac{H_s(3c/2, N+2-2c)}{4^s s!}.$$

Finally, we construct rational **de Rham** matrices, with elements

$$D_N(a,b) \equiv \sum_{c=-b}^{a} d_{a-c}(N,-c)d_{b+c}(N,c)c^{N+1}.$$

Summary

1. Chan-Zudilin transformations improve computational efficiency.
2. After 10 years, all conjectures on 5-Bessel moments have been proven.
3. The L-series for 6 and 8 Bessel functions are modular.
4. An understanding of Bessel moments as quasi-periods is emerging.
5. Relations between determinants of Feynman integrals and L-series have been discovered up to 22 loops and presumably go on for ever.
6. There are quadratic relations of the form $P_N D_N P_N^{\mathrm{tr}} = B_N$ with period, de Rham and Betti matrices that have been specified.

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Edinburgh, Copenhagen, Vienna, Zeuthen, Southampton, Durham, Bonn,
דוד $\sim$ ΔϜΔ $\sim$ $\eta^{24}\,\psi\,\eta^{24}$
went out whithersoever Saul sent him.
(1 Samuel 18:5.)