ntro: $\zeta(k) = \sum_{n\geq 1} \frac{1}{n^k}$, $k \geq 2$, $\zeta(1) := 0$

$\zeta(r,s) = \sum_{m>n>0} \frac{1}{m^r n^s}$, $r\geq 2$, $s \geq 1$, $\zeta(1,1) := 0$, $\zeta(1,s) := -\zeta(s,1) - \zeta(s+1)$ if $s \geq$

$Z^{(1)}(X_1) := \sum_{k\geq 1} \zeta(k) X_1^{k-1} \in \mathbb{R}[[X_1]]$, $Z^{(2)}(X_1,X_2) := \sum_{r,s\geq 1} \zeta(r,s) X_1^{r-1} X_2^{s-1} \in \mathbb{R}[[X_1,X_2]]$

roposition (Euler): The pair $(Z^{(1)}, Z^{(2)})$ satisfies

(1) $Z^{(1)}(X_1) Z^{(1)}(X_2) = Z^{(2)}(X_1+X_2, X_1) + Z^{(2)}(X_1+X_2, X_2)$

(2) $Z^{(1)}(X_1) Z^{(1)}(X_2) = Z^{(2)}_*(X_1,X_2) + Z^{(2)}_*(X_2,X_1) + \frac{Z^{(1)}(X_1) - Z^{(1)}(X_2)}{X_1 - X_2}$

where $Z^{(2)}_* = Z^{(2)} + \frac{\mu^2}{48}$, $\mu = (2\pi i)$

("(extended) double shuffle relations")

· Are there solutions to (1),(2) with rational coefficients ($\mu=1$)?

$Z^{(1)}_{\mathbb{Q}}(X_1) = \sum_{k\geq 1} \zeta_{\mathbb{Q}}(k) X_1^{k-1}$, $Z^{(2)}_{\mathbb{Q}}(X_1,X_2) = \sum_{r,s\geq 1} \zeta_{\mathbb{Q}}(r,s) X_1^{r-1} X_2^{s-2}$

depth 1: Know: $\zeta(2k) = -\frac{B_{2k}(2\pi i)^{2k}}{2(2k)!}$ (Euler)

Conjecture: $\mathbb{Q}[2\pi i] \cap \mathbb{Q}[\zeta(3), \zeta(5), \zeta(7), \ldots] = \mathbb{Q}$

· Set $\zeta_{\mathbb{Q}}(k) = \begin{cases} -\frac{B_k}{2k!} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

Then $Z^{(1)}_{\mathbb{Q}}(X_1) = \underbrace{-\frac{1}{2}\left(\frac{1}{e^{X_1}-1} + \frac{1}{2}\right)}_{B^{(1)}(X_1)} + \underbrace{\frac{1}{2X_1}}_{P^{(1)}(X_1)}$

pth 2: Notation: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{Z})$, $p \in \mathbb{Q}[[X_1,X_2]]$

$p(X_1,X_2)|M := p(aX_1+bX_2, cX_1+dX_2)$

· Set $B^{(2)}(X_1,X_2) := \frac{1}{3}\left(B^{(1)}(X_1)B^{(1)}(X_2)|(1-U^2) - \frac{1}{32}\right)$, $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$P^{(2)}(X_1,X_2) = \frac{1}{3} P^{(1)}(X_1) P^{(1)}(X_2) |(1-U)$

osition: The pair $(Z^{(1)}_{\mathbb{Q}}, S^{(2)})$, where

$S^{(2)}(X_1,X_2) = B^{(2)}(X_1,X_2) + P^{(2)}(X_1,X_2) + B^{(1)}(X_1)P^{(1)}(X_2)|(1-\varepsilon U - U^2)$, $\varepsilon = ($

satisfies (1),(2) above ($\mu = 1$).

Key point: $\beta^{(1)}(X_1)\beta^{(1)}(X_2)|(\cancel{4}1+S) = \beta^{(1)}(X_1)\beta^{(1)}(X_2)|(1+U+U^2) = 0$, $S = (1 \ \ 0)$

$B^{(1)}(X_1)B^{(1)}(X_2)|(1+S) = 0$, $B^{(1)}(X_1)B^{(1)}(X_2)|(1+U+U^2) = \frac{1}{16}$

Problem: $S^{(2)}$ has poles, i.e. $S^{(2)} \notin \mathbb{Q}[[X_1, X_2]]$.

- $G = \langle U, \varepsilon \rangle \cancel{\cong} S_3 \subseteq PGL_2(\mathbb{Z})$, $G_0 \cong S_3$
- $C^{(2)}(X_1, X_2) = \frac{1}{|G|} \sum_{g \in G} \det(g) \cdot B^{(1)}(X_1)\beta^{(1)}(X_2)|g$

Proposition: i) $Z_{\mathbb{Q}}^{(2)} := S^{(2)} + C^{(2)} \in \mathbb{Q}[[X_1, X_2]]$

ii) $(Z_{\mathbb{Q}}^{(1)}, Z_{\mathbb{Q}}^{(2)})$ satisfies (1),(2) above.

Remarks: i) Construction of $Z_{\mathbb{Q}}^{(2)}$ goes back to Ecalle ("zig I") (and (independently) Gangl-Kaneko-Zagier ("Bernoulli realization")

- $B^{(1)}(X_1)B^{(1)}(X_2)$ is essentially the gen. series of odd period polynomials of Eisenstein series

ii) $Z_{\mathbb{Q}}^{(2)}$ is <u>not</u> unique! Can add any solution to linearized double shuffle equations in depth 2.

iii) Similar construction in depth three (Ecalle, Brown)