The modular forms of the simplest quantum field theory

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Abstract

Much of the $(2,5)$ minimal model in conformal field theory is described by very classical mathematics: Schwarz' work on algebraic hypergeometric functions, Klein's work on the icosahedron, the Rogers-Ramanujan functions etc. Unexplored directions promise equally beautiful results.

1 The $(2,5)$ MM for $g = 1$

1.1 Some ODEs

For $g = 1$, the 1-point function $\langle T(z)\rangle$ is constant in position, denoted by $\langle \mathbf{T}\rangle$. For the $(2,5)$ minimal model, the Virasoro OPE is given by

$$T(z)\otimes T(0)\mapsto \frac{c/2}{z^4}.1+\frac{1}{z^2}\{T(z)+T(0)\}-\frac{1}{5}T''(0)+O(z)\,,$$

where $c=-22/5$. This implies the 2-point function

$$\langle T(z)T(0)\rangle=\frac{c}{2}\wp^2(z|\tau)\langle\mathbf{1}\rangle+2\wp(z|\tau)\langle\mathbf{T}\rangle-c\frac{\pi^4}{15}E_4\langle\mathbf{1}\rangle\,.$$

Changes in the modulus τ are generated by the Virasoro field T. We obtain a system of ODEs of the type studied by [7] and [2]:

$$\begin{aligned}\frac{1}{2\pi i}\frac{d}{d\tau}\langle\mathbf{1}\rangle &= \oint\langle T(z)\rangle\,\frac{dz}{(2\pi i)^2}=\frac{1}{(2\pi i)^2}\langle\mathbf{T}\rangle\\ \frac{1}{2\pi i}\frac{d}{d\tau}\langle\mathbf{T}\rangle &= \oint\langle T(w)T(z)\rangle\,\frac{dz}{(2\pi i\,)^2}=\frac{1}{6}E_2\langle\mathbf{T}\rangle+\frac{11}{3600}(2i\pi)^2E_4\langle\mathbf{1}\rangle\,,\end{aligned}$$

or in terms of the Serre derivative $\mathfrak{D}:=\frac{1}{2\pi i}\frac{d}{d\tau}-\frac{k}{6}E_2$ (defined on modular forms of weight $2k$),

$$\mathfrak{D}^2\langle\mathbf{1}\rangle=\frac{11}{3600}\,E_4\langle\mathbf{1}\rangle\,. \tag{1}$$

Its solutions are the 0-point functions named after Rogers-Ramanujan (in the following referred to as RR)

$$\langle\mathbf{1}\rangle_i=q^{-\frac{c}{24}}\chi_i\,,\quad i=1,2\,,$$

for the characters

$$\chi_1 = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + \ldots \quad \text{(vacuum)}$$
$$\chi_2 = q^{-\frac{1}{5}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = q^{-\frac{1}{5}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + \ldots\right) .$$

Here $q = \exp(2\pi i\tau)$. The partition function is

$$Z = |\langle \mathbf{1} \rangle_1|^2 + |\langle \mathbf{1} \rangle_2|^2 .$$

The space of all fields factorises as

$$F = F_V \otimes \overline{F_V} \oplus F_W \otimes \overline{F_W} ,$$

where F_V and F_W denote the space of holomorphic fields (irreps of the Virasoro algebra) that correspond to states in V and W, respectively, and the bar marks complex conjugation.

We shall use the algebraic description of the torus as a double cover $\mathbb{P}^1_{\mathbb{C}}$ defined by

$$y^2 = x(x-1)(x-\lambda) ,$$

where $\lambda \in \mathbb{C}$ is the squared Jacobi modulus. Describing the change of λ by an action of $T(x)$, we find the ODE

$$\frac{d^2}{d\lambda^2} f + p \frac{d}{d\lambda} f + q f = 0 \tag{2}$$

with rational coefficients

$$q = \frac{-\alpha\beta}{\lambda(1-\lambda)} , \quad p = \frac{\gamma}{\lambda} + \frac{\gamma - (\alpha + \beta + 1)}{1-\lambda} ,$$

where

$$(\alpha, \beta; \gamma) = \left(\frac{7}{10}, \frac{11}{10}; \frac{7}{5}\right) \quad \text{or} \quad \left(\frac{3}{10}, -\frac{1}{10}; \frac{2}{5}\right) .$$

Its equivalence to (1) can be seen from [4]

$$\frac{d\lambda}{1-\lambda} = \pi i\, E_2 d\tau - 6 d(\log \ell) ,$$

where ℓ is the inverse length of the real period.

Comparison of the two approaches yields

$$\langle \mathbf{1} \rangle_1 = [\lambda(\lambda-1)]^{-c/24} \, {}_2F_1\left(\frac{7}{10}, \frac{11}{10}; \frac{7}{5}; \lambda\right)$$
$$\langle \mathbf{1} \rangle_2 = [\lambda(\lambda-1)]^{-1/5-c/24} \, {}_2F_1\left(\frac{3}{10}, -\frac{1}{10}; \frac{2}{5}; \lambda\right) .$$

These relations seem to be new though they're closely related to Schwarz' work [8] as will be indicated in the rest of this section.

1.2 Algebraicity of solutions to the hypergeometric DE [8]

From general theory, we know that RR are algebraic [10]. Yet, using differential equations, we hope to generalise some of the classical arguments to higher genus.

A necessary condition for the general solution of the hypergeometric differential equation (2) to be algebraic in λ is that $\alpha, \beta, \gamma \in \mathbb{Q}$ (Kummer), which we will assume in the following.

Claim 1. *Let f_1, f_2 be solutions of (2), for some choice of $\alpha, \beta, \gamma \in \mathbb{Q}$, such that f_1/f_2 is algebraic. Then f_1, f_2 are themselves algebraic.*

Proof. (Heine) Let $s = f_1/f_2$. Since

$$s' =: \frac{W}{f_2^2} \, ,$$

it suffices to show that the Wronskian $W = f_1' f_2 - f_2' f_1$ is algebraic: We have

$$W' = f_1'' f_2 - f_2'' f_1 = -pW$$

by eq. (2), so

$$W \sim \exp\left(-\int p \, d\lambda\right) = \lambda^A (\lambda - 1)^B \, ,$$

where by assumption $A, B \in \mathbb{Q}$. □

Outline: Given two independent algebraic solutions f_1, f_2 to (2), their quotient $s = f_1/f_2$ solves a linear third order differential equation in λ [8]. By linearity of (2), s is invariant under Möbius transformations. s defines a map

$$\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\} \;\to\; \mathbb{P}^1_{\mathbb{C}} \, , \quad \lambda \mapsto (f_1 : f_2) \, . \tag{3}$$

Suppose $f_1^{\mathbb{R}}, f_2^{\mathbb{R}}$ are real on $(0,1)$ and their quotient $s^{\mathbb{R}} = f_1^{\mathbb{R}}/f_2^{\mathbb{R}}$ maps the interval onto a segment $I_{(0,1)}$ of $\mathbb{P}^1_{\mathbb{R}}$. Via an analytic extension to H^+, $s^{\mathbb{R}}$ can be extended to the intervals $(-\infty, 0)$ and $(1, \infty)$. An interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ is mapped to two arcs forming some angle. Together, the images of $(0,1)$, $(1,\infty)$ and $(-\infty, 0)$ form a triangle in $\mathbb{P}^1_{\mathbb{C}}$. In the elliptic case (angular sum $> 180°$), the triangle is conformally equivalent to a spherical triangle on S^2 whose edges are formed by arcs of great circles. By crossing any of the intervals $(1,\infty)$ $(0,1)$, or $(-\infty, 0)$, $s^{\mathbb{R}}$ can be further continued to $\mathbb{H}^-$. The reflection symmetry w.r.t. the real line in the λ-plane corresponds to circle inversion w.r.t. the respective triangle edge.

Analytic continuating along paths circling the singularities in any order may in general produce an infinite number of triangles in $\mathbb{P}^1_{\mathbb{C}}$. The number is finite if and only if the quotient of solutions is algebraic.

The problem is therefore transformed into sorting out all spherical triangles whose symmetric and congruent repetitions lead to a finite number only of triangles of different shape and position.

A necessary condition for a spherical shape and its symmetric and congruent repetions to form a closed Riemann surface is that the edges lie in planes which are symmetry planes of a regular polytope. For the spherical triangles, this leads to a finite list of triples of angles that correspond to platonic solids.

1.3 Invariants related to the icosahedron [3]

Theorem 1. *The function*

$$e^{-\pi i/5}\frac{\theta\begin{bmatrix}3/5\\1\end{bmatrix}(5\tau)}{\theta\begin{bmatrix}1/5\\1\end{bmatrix}(5\tau)} = e^{2\pi i\tau/5}\frac{\sum_{n=-\infty}^{\infty}(-1)^n(e^{\pi i\tau})^{5n^2+3n}}{\sum_{n=-\infty}^{\infty}(-1)^n(e^{\pi i\tau})^{5n^2+n}}$$

is algebraic.

To prove this statement, consider an icosahedron (symmetry group A_5) inscribed in the unit sphere. Subdivide each face into 6 triangles by connecting its centroid with the surrounding vertices and edge midpoints. Projecting the vertices and edges onto the sphere from its center results in a tessellation of the sphere by triangles.

Let $\tilde{V}$ be the degree 12 invariant homogeneous poynomial in projective coordinates s_1, s_2 on $\mathbb{P}^1_{\mathbb{C}}$ which has a simple root at $s_1 s_2 = 0$. (We assume here that the two poles are vertices). The Hessian $\tilde{F}$ of $\tilde{V}$ and the functional determinant of $\tilde{V}$ and $\tilde{F}$ have roots corresponding to 20 face centers and to 30 mid-edge points of the ikosahedron, respectively. The three polynomials satisfy a syzygy. By stereographic projection, the tessellation of the sphere gives rise to a configuration of intersecting arcs on the extended complex plane $\hat{\mathbb{C}}$ (with coordinate z). We call a point of intersection an edge point, a face point or a vertex depending on its origin in the icosahedron. The 11 finite vertices and the face points define simple roots of monic polynomials V and F of degree 11 and 12 respectively. To A_5 corresponds the subgroup $G_{60} \subseteq PSL(2,\mathbb{C})$ of Möbius transformations. F, V transform under G_{60} in such a way that

$$J(z) = \frac{F^3(z)}{V^5(z)}$$

is invariant. For $\tilde{z} = z^5$, the associated icosahedral equation reads

$$(\tilde{z}^4 - 228\tilde{z}^3 + 494\tilde{z}^2 + 228\tilde{z} + 1)^3 + \tilde{z}(\tilde{z}^2 + 11\tilde{z} - 1)^5 J(z) = 0\,.$$

When $J(z)$ equals the classical j-invariant

$$j = 2^8\frac{(1-\lambda(1-\lambda))^3}{\lambda^2(1-\lambda)^2}$$

a solution is given by $\tilde{z} = z^5$ where z is the function in the claim. The icosahedral equation for the modular j-invariant is the minimal polynomial of z^5 over $\mathbb{Q}(j(\tau))$.

1.4 Discussion of the RR case

As mentioned abbove, we know that RR are algebraic.

Claim 2. *The pair of RR $\langle \mathbf{1}\rangle_1, \langle \mathbf{1}\rangle_2)$ defines a finite covering of $\mathbb{P}^1_{\mathbb{C}}$.*

Indeed, we have a map

$$RR: \quad \mathbb{H}^+ \to \mathbb{P}^1_{\mathbb{C}}, \quad \tau \mapsto (\langle \mathbf{1}\rangle_1 : \langle \mathbf{1}\rangle_2)$$

since RR are nowhere simultaneously zero. The map descends to a map

$$\Gamma \setminus \mathbb{H}^+ \;\to\; \mathbb{P}^1_{\mathbb{C}}\,,$$

for some finite index subgroup $\Gamma \subset \Gamma_1 = SL(2,\mathbb{Z})$. The fundamental domain $\mathcal{F}_\Gamma$ of Γ is a finite union of copies of the fundamental domain $\mathcal{F}_1$ of Γ_1. When compactified, the latter is conformally equivalent to $\mathbb{P}^1_{\mathbb{C}}$, and $\mathcal{F}_\Gamma$ defines a finite cover of $\mathbb{P}^1_{\mathbb{C}}$.

Claim 3. *The quotient $r(\tau) = \frac{\langle \mathbf{1}\rangle_1}{\langle \mathbf{1}\rangle_2}$ is algebraic. r^5 defines a 12-fold covering of $\mathbb{P}^1_{\mathbb{C}}$.*

Indeed, $r(\tau)$ is modular on $\Gamma(5)$, the principal congruence subgroup of level 5. The modular curve

$$\Gamma(5) \setminus (\mathbb{H}^+ \cup \{\infty\})$$

has $g = 0$ and the symmetry of an icosahedron A_5. One verifies [1] that $r(\tau)$ equals the function studied in Claim 1.

Observe that also $d(\log r^5) = \frac{5W}{\langle \mathbf{1}\rangle_1 \langle \mathbf{1}\rangle_2}$ defines a 12-fold covering of $\mathbb{P}^1_{\mathbb{C}}$, where the Wronskian W is known explicitely from the hypergeometric equation (2).

The RR are themselves algebraic; they feature as the most symmetric case (no. XI, i.e. all three angles equal $2\pi/5$) in the list of Schwarz [8].

Functions that satisfy a hypergeometric differential equation for particular values of α, β, γ are algebraic (Schwarz), thus they define a finite covering of $\mathbb{P}^1_{\mathbb{C}}$ (Riemann). This leads to the consideration of platonic solids. Using differential equations, these methods should generalise to higher genus. Thus we expect that the 0-point functions of the $(2,5)$ minimal model continue to be algebraic for $g \geq 2$.

2 The $(2,5)$ MM for $g \geq 2$

2.1 ODEs for the 0-point functions

Theorem 2. *[5]*
0-point functions for genus 2 solve a 5th order linear ODE (w.r.t. any of its ramification points) with regular singularities. (The system is given explicitely.)

The main idea of the proof is to use algebraic coordinates, $x = \wp(z|\tau)$, $y = \partial_z \wp(z|\tau)$.

Alternatively: Use the rich theory of elliptic functions by letting the genus $g = 2$ surface degenerate (Deligne-Mumford compactification (1969) of the moduli space of Riemann surfaces).

For $i = 1, 2$, let (Σ_i, P_i) with $P_i \in \Sigma_i$ be a non-singular Riemann surface of genus g_i with puncture P_i. Let z_i be a local coordinate vanishing at P_i. Excise sufficiently small discs $\{|z_1| < \varepsilon\}$ and $\{|z_2| < \varepsilon\}$ from Σ_1 and Σ_2, respectively, and sew the two remaining surfaces by the condition

$$z_1 z_2 = \varepsilon^2 \tag{4}$$

on tubular neigbourhoods of the circles $\{|z_i| = \varepsilon\}$.

This operation yields a non-singular Riemann surface of genus $g_1 + g_2$ with no punctures. The $g = 2$ partition function is obtained perturbatively as a power expansion in ε. Two possibilities [9]:

1. The $g = 2$ surface decomposes into **two tori** (with modulus τ and $\hat{\tau}$, respectively) when three ramification points run together (cutting through the neck along a cycle that is homologous to zero). We have for $a, b \in \{1, 2\}$ (Gilroy & Tuite, and [6]),

$$\langle 1\rangle_{a,b}^{g=2}(q,\hat{q},\varepsilon) = \langle 1\rangle_a \widehat{\langle 1\rangle}_b - \frac{2}{c}\varepsilon^2 \langle T\rangle_a \widehat{\langle T\rangle}_b - \frac{7}{31c}\varepsilon^6 \langle L_4 L_2 1\rangle_a \widehat{\langle L_4 L_2 1\rangle}_b + O(\varepsilon^8) .$$

(The hat refers to the modulus $\hat{\tau}$.) In addition there is a 5th solution:

$$\langle 1\rangle_{\varphi}^{g=2}(q,\hat{q},\varepsilon) = \varepsilon^{-1/5}\left(\eta\,\widehat{\eta}\right)^{-2/5}\left\{1 + \frac{13}{8,208,000}(2\pi i)^8 \varepsilon^4 E_4 \widehat{E}_4 + O(\varepsilon^6)\right\} .$$

2. One obtains a **single torus** by letting two ramification points run together (cutting through a genus $g = 2$ surface along a cycle that is not homologous to zero) [6].

The invariant is

$$Z^{g=2} = \sum_{i}^{4} |\langle \mathbf{1}\rangle_i^{g=2}|^2 + \kappa |\langle \mathbf{1}\rangle_5^{g=2}|^2 ,$$

for some $\kappa \in \mathbb{R}$ (so that Z is modular).

2.2 Related ODEs

The physical interpretation of the 5th solution requires another field Φ with (locally) $\Phi = \varphi_{\text{hol}} \otimes \overline{\varphi}_{\text{hol}}$ where $\varphi := \varphi_{\text{hol}}$ is lowest weight vector (of weight $-1/5$) in the irrep F_W of the Virasoro algebra. Its 1-point function satisfies

$$\mathcal{D}\langle\varphi\rangle = 0 .$$

Thus

$$\langle\varphi\rangle = \eta^{-2/5} = q^{-\frac{1}{60}} \prod_{n\geq 1}(1-q^n)^{-2/5}$$

We apply the methods used previoulsy in F_V to $\varphi \in F_W$. The continuation of the theory from $g = 1$ to $g = 2$ requires both lowest weight vectors.

Claim 4. *The 2-pt function of φ satisfies a 3rd order ODE with regular singularities,*

$$\frac{25}{12}\langle\varphi^{(3)}(z)\varphi(0)\rangle = h\wp'(z)\langle\varphi(z)\varphi(0)\rangle + \wp(z)\langle\varphi'(z)\varphi(0)\rangle .$$

In algebraic coordinates (and the corresponding field $\check{\varphi}$), the above ODE reads

$$y^{4/5}\left(p(x)\frac{d^3}{dx^3} + f(x)\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x)\right)\Psi(x) = 0 ,$$

where $\Psi(x) = \langle\check{\varphi}(x)\varphi(0)\rangle$,

$$p(x) = 4\left(x^3 - \frac{\pi^4}{3}E_4 x - \frac{2}{27}\pi^6 E_6\right) ,$$

and

$$
\begin{aligned}
f &= \frac{6}{5}p' \\
g &= \frac{3}{100}\frac{[p']^2}{p} + \frac{9}{50}p'' \\
h &= -\frac{33}{500}\frac{[p']^3}{p^2} + \frac{33}{250}\frac{p'p''}{p} - \frac{288}{125}\,.
\end{aligned}
$$

In particular, the ODE has simple poles at the four ramification points.

A proof of the statement can be found in [6].

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