Multiple Zeta Values and Alternating Multiple Zeta Values Arising from a Combinatorial Problem

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Outline

Introduction

Most of this talk is about joint work with Guy Louchard, Markus Kuba, and Moti Levy. From my point of view the interesting part is about multiple zeta and alternating multiple zeta values, so I will establish notation for them first. For positive integers $i_1, \ldots, i_k$ with $i_1 > 1$, the corresponding multiple zeta value is the real number

$$\zeta(i_1, i_2, \ldots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

This can be generalized to alternating multiple zeta values with signs in the numerator, as in

$$\zeta(\bar{2}, 3, \bar{1}) = \sum_{n_1 > n_2 > n_3 \geq 1} \frac{(-1)^{n_1+n_3}}{n_1^2 n_2^3 n_3}.$$

Introduction cont'd

An alternating multiple zeta value converges unless the first entry is an unbarred 1; for example, $\zeta(\bar{1}) = -\log 2$ and $\zeta(\bar{n}) = (1-2^{1-n})\zeta(n)$ for $n \geq 2$.
Now for the combinatorial problem: consider the integral

$$I(n) = \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}}\, dx.$$

Then $I(n)$ represents, for example, the expected value of the random variable $(U^n + (1-U)^n)^{\frac{1}{n}}$, where U is uniform on $[0,1]$. Guy Louchard sought an asymptotic series

$$I(n) = I_0 + \frac{I_1}{n} + \frac{I_2}{n^2} + \frac{I_3}{n^3} + \cdots$$

Introduction cont'd

Because of the symmetry in $x = \frac{1}{2}$, the integral can be written

$$I(n) = 2\int_0^{\frac{1}{2}} (1-x)\left(1 + \left(\frac{x}{1-x}\right)^n\right)^{\frac{1}{n}} dx$$

At this point one can set $x = \frac{1}{2} - \frac{u}{4n}$ and write the integral as

$$I(n) = \frac{1}{2n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\left[1 + \left(\frac{1-\frac{u}{2n}}{1+\frac{u}{2n}}\right)^n\right]^{\frac{1}{n}} du.$$

After several more algebraic steps $I(n)$ is

$$\frac{1}{2n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\left(1 + \frac{\log(1+e^{-u})}{n} + \frac{\log(1+e^{-u})^2}{2n^2} - \frac{u^3 e^{-u}}{12n^3(1+e^{-u})} + \frac{\log(1+e^{-u})^3}{6n^3} + \cdots\right) du$$

Introduction cont'd

The first term can be separated out as

$$\frac{1}{2n}\int_0^{2n}\left(\frac{1}{2}+\frac{u}{4n}\right)du=\frac{3}{4},$$

and the remainder written in the form

$$\int_0^{2n}\left(\frac{1}{2}+\frac{u}{4n}\right)\left(\frac{\log(1+e^{-u})}{2n^2}+\frac{\log(1+e^{-u})^2}{4n^3}\right.$$
$$\left.-\frac{u^3e^{-u}}{24n^4(1+e^{-u})}+\frac{\log(1+e^{-u})^3}{12n^4}+\cdots\right)du$$

which is expanded in powers of $\frac{1}{n}$ and the integral taken from 0 to ∞.

Introduction cont'd

This gives $I_0 = \frac{3}{4}$, $I_1 = 0$,

$$I_2 = \frac{1}{4}\int_0^\infty \log(1+e^{-u})du = \frac{1}{8}\zeta(2),$$

$$I_3 = \frac{1}{8}\int_0^\infty \left(u\log(1+e^{-u}) + \log(1+e^{-u})^2\right) du = \frac{1}{8}[-\zeta(\bar{3}) + \zeta(\bar{2},1)] = \frac{1}{8}\zeta(3),$$

and

$$I_4 = \frac{1}{48}\int_0^\infty \left(-\frac{u^3e^{-u}}{1+e^{-u}} + 3u\log(1+e^{-u})^2 + 2\log(1+e^{-u})^3\right)du = \frac{1}{8}[\zeta(\bar{4})+\zeta(\bar{3},1)-2\zeta(\bar{2},1,1)] = -\frac{3}{32}\zeta(4).$$

Introduction, cont'd

Above we used several results expressing integrals in terms of alternating multiple zeta values, which we won't discuss since we're about to use a more general method. With more persistence (and four more terms) Guy obtained

$$I_5 = \frac{1}{8}[2\zeta(\bar{5}) - \zeta(\bar{4},1) - \zeta(\bar{3},1,1) + 2\zeta(\bar{2},1,1,1)] = -\frac{1}{8}\zeta(2)\zeta(3)$$

and I computed I_6 as

$$\frac{1}{8}[-2\zeta(\bar{6}) - 2\zeta(\bar{5},1) + \zeta(\bar{4},1,1) + \zeta(\bar{3},1,1,1) - 2\zeta(\bar{2},1,1,1,1)] \\ = \frac{1}{8}\left[\frac{83}{32}\zeta(6) - \frac{1}{2}\zeta(3)^2\right].$$

Introduction cont'd

What's notable is the apparent regularity of the pattern when I_k is written as a sum of alternating multiple zeta values, and the apparent fact that the expression can be reduced to a rational polynomial in the ordinary zeta values $\zeta(i)$. Although it's true that any multiple zeta value of weight ≤ 7 is a rational polynomial in the $\zeta(i)$, with alternating multiple zeta values things go wrong much sooner: $\zeta(\bar{3}, 1)$ appears to have no rational expression in terms of ordinary zeta values. It seems that all the "exotic" terms in the linear combination of alternating multiple zeta values that expresses I_k are cancelling out. In fact both "apparent" statements are true, as we shall see.

A better substitution

There is in fact a better substitution to expand out $2\int_0^{\frac{1}{2}}(1-x)\left[1+\left(\frac{x}{1-x}\right)^n\right]^{\frac{1}{n}}dx$ than what was used above. Let $u=\frac{x}{1-x}$, so that the integral becomes

$$2\int_0^1 (1+u^n)^{\frac{1}{n}}\frac{du}{(1+u)^3} = \\ 2\int_0^1 \left(1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{1}{n}\log(1+u^n)\right)^k\right)\frac{du}{(1+u)^3}$$

Now use

$$(\log(1+x))^k = k!\sum_{m=1}^{\infty}\frac{x^m}{m!}s(m,k)$$

where the $s(m,k)$ are (signed) Stirling numbers of the first kind.

Expanding into alternating MZVs

Hence

$$I(n) = 2\int_0^1 \frac{du}{(1+u)^3} + 2\sum_{k=1}^{\infty}\int_0^1 \frac{u^{nm}s(m,k)}{m!n^k}\frac{du}{(1+u)^3}$$
$$= \frac{3}{4} + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{s(m,k)}{m!n^k}\int_0^1 \frac{u^{mn}}{(1+u)^3}du.$$

If we define truncated multiple zeta values by

$$\zeta_r(i_1,\dots,i_k) = \sum_{r\ge n_1>\cdots>n_k\ge 1}\frac{1}{n_1^{i_1}\cdots n_k^{i_k}},$$

then it is not hard to see that

$$s(m,k) = (-1)^{m-k}(m-1)!\zeta_{m-1}(\{1\}_{k-1})$$

where $\{1\}_n$ means 1 repeated n times.

Expanding into alternating MZVs cont'd

Thus $I(n)$ is

$$\frac{3}{4} + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-k}\zeta_{m-1}(\{1\}_{k-1})}{n^k m}\int_0^1 \frac{u^{mn}}{(1+u)^3}du.$$

Then if

$$\int_0^1 \frac{u^r}{(1+u)^3}du = \sum_{j=1}^{\infty}\frac{\beta_{j-1}}{r^j}$$

This becomes

$$I(n) = \frac{3}{4} + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{m-k}\zeta_{m-1}(\{1\}_{k-1})\beta_{j-1}}{n^{k+j}m^{j+1}}$$

Expanding into alternating MZVs cont'd

or

$$I(n) = \frac{3}{4} + 2\sum_{p=2}^{\infty}\frac{1}{n^p}\sum_{m=1}^{\infty}\sum_{k=1}^{p-1}(-1)^{m-k}\frac{\zeta_{m-1}(\{1\}_{k-1})\beta_{p-k-1}}{m^{p-k-1}}$$

$$= \frac{3}{4} + 2\sum_{p=2}^{\infty}\frac{1}{n^p}\sum_{k=1}^{p-1}(-1)^k\beta_{p-k-1}\zeta(\overline{p-k-1},\{1\}_{k-1})$$

Thus $I_0 = \frac{3}{4}$, $I_1 = 0$, and for $p \geq 2$,

$$I_p = 2\sum_{k=1}^{p-1}(-1)^k\beta_{p-k-1}\zeta(\overline{p-k-1},\{1\}_{k-1})$$

$$= 2\sum_{j=2}^{p}(-1)^{p-j-1}\beta_{j-2}\zeta(\bar{j},\{1\}_{p-j}).$$

Finding the β_j

It remains to find the β_j.

Lemma

For $j \geq 0$, $\beta_j = \frac{(-1)^j}{4}(E_{j+1}(-1) + E_{j+2}(-1))$, where $E_n(x)$ is the nth Euler polynomial.

To prove this, first make the change of variable $u = e^{-t}$ to get

$$\int_0^1 \frac{u^r}{(1+u)^3} du = \int_0^\infty \frac{e^{-t}}{(1+e^{-t})^3} e^{-rt} dt.$$

Now by direct computation

$$\frac{e^{-t}}{(1+e^{-t})^3} = \frac{1}{4}\left[\frac{d^2}{dt^2}\left(\frac{2e^t}{1+e^{-t}}\right) - \frac{d}{dt}\left(\frac{2e^t}{1+e^{-t}}\right)\right]$$

Finding the β_j cont'd

The generating function for the Euler polynomials is

$$\mathcal{E}(t,x) = \frac{2e^{tx}}{1+e^t} = \sum_{j=0}^{\infty} E_j(x)\frac{t^j}{j!}.$$

Differentiation of $\mathcal{E}(-t,-1)$ gives

$$\frac{d}{dt}\left(\frac{2e^t}{1+e^{-t}}\right) = -\sum_{n=0}^{\infty}(-1)^n E_{n+1}(-1)\frac{t^n}{n!}$$

$$\frac{d^2}{dt^2}\left(\frac{2e^t}{1+e^{-t}}\right) = \sum_{n=0}^{\infty}(-1)^n E_{n+2}(-1)\frac{t^n}{n!}$$

so that

Finding the β_j cont'd

$$\int_0^\infty \frac{e^{-t}}{(1+e^{-t})^3} e^{-rt}dt =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4}(E_{n+2}(-1)+E_{n+1}(-1)) \int_0^\infty \frac{t^n}{n!}e^{-rt}dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4}(E_{n+2}(-1)+E_{n+1}(-1))\frac{1}{r^{n+1}}$$

and the conclusion follows. Hence for $p \geq 2$,

$$I_p = \frac{(-1)^{p-1}}{2}\sum_{j=2}^{p}(E_{j-1}(-1)+E_j(-1))\zeta(\bar{j},\{1\}_{p-j}).$$

Euler polynomials

We can simplify this further using properties of Euler polynomials. Set $x = -1$ in the identity

$$E_n(x) + E_n(x+1) = 2x^n$$

to get $E_n(-1) = 2(-1)^n - E_n(0)$, and thus

$$E_{j-1}(-1) + E_j(-1) = -E_{j-1}(0) - E_j(0) = \begin{cases} -E_j(0), & j \text{ odd}, \\ -E_{j-1}(0), & j \text{ even}, \end{cases}$$

since $E_n(0) = 0$ for $n > 0$ even. We have shown

Theorem

Let $a_n = \frac{1}{2}E_{2n+1}(0)$. Then for $p \geq 2$,

$$I_p = (-1)^p \sum_{j=2}^{p} a_{\lfloor \frac{j-1}{2} \rfloor} \zeta(\bar{j}, \{1\}_{p-j}).$$

Euler polynomials cont'd

The numbers a_n have the generating function

$$\sum_{n=0}^{\infty} a_n \frac{t^{2n+1}}{(2n+1)!} = -\frac{1}{2}\tanh\frac{t}{2}$$

and the expression

$$a_n = \frac{(1-2^{2n+2})B_{2n+2}}{2n+2}$$

in terms of Bernoulli numbers. The first few are

$$a_0 = -\frac{1}{4},\ a_1 = \frac{1}{2},\ a_2 = -\frac{1}{4},\ a_3 = \frac{17}{16},\ a_4 = -\frac{31}{4},\ a_5 = \frac{691}{8}.$$

Expanding into alternating MZVs

Thus, for example,

$$
\begin{aligned}
I_{11} = {} & \frac{1}{4}\zeta(\bar{2},\{1\}_9) - \frac{1}{8}\zeta(\bar{3},\{1\}_8) - \frac{1}{8}\zeta(\bar{4},\{1\}_7) + \frac{1}{4}\zeta(\bar{5},\{1\}_6) \\
& + \frac{1}{4}\zeta(\bar{6},\{1\}_8) - \frac{17}{16}\zeta(\bar{7},\{1\}_4) - \frac{17}{16}\zeta(\bar{8},\{1\}_3) + \frac{31}{4}\zeta(\bar{9},1,1) \\
& + \frac{31}{4}\zeta(\overline{10},1) - \frac{691}{8}\zeta(\overline{11}).
\end{aligned}
$$

Using known identities, this reduces to

$$
\begin{aligned}
& \frac{63}{8}\zeta(11) + \frac{58007}{3072}\zeta(3)\zeta(8) + \frac{5187}{256}\zeta(5)\zeta(6) + \frac{135}{8}\zeta(4)\zeta(7) \\
& + \frac{115}{12}\zeta(2)\zeta(9) - \frac{13}{48}\zeta(2)\zeta(3)^3 - \frac{21}{32}\zeta(3)^2\zeta(5)
\end{aligned}
$$

which is no accident.

Kölbig's results

In the 1980s, K. S. Kölbig wrote several papers about the function

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1}(t)\log^p(1-zt)}{t} dt,$$

and in particular about its values at $z = \pm 1$. These are relevant to our subject because of the following result.

Lemma

If $|z| \leq 1$, then

$$S_{n,p}(z) = \sum_{j_1 > j_2 > \cdots > j_p \geq 1} \frac{z^{j_1}}{j_1^{n+1} j_2 \cdots j_p}$$

Proof of lemma

Since

$$\log(1-zt) = -\sum_{i\geq 1} \frac{z^i t^i}{i}, \quad \int_0^1 t^{m-1}\log^{n-1}(t)dt = \frac{(n-1)!}{m^n}$$

we have

$$\int_0^1 \frac{\log^{n-1}(t)\log^p(1-zt)}{t}dt =$$

$$(-1)^p \sum_{i_1=1}^{\infty}\cdots\sum_{i_p=1}^{\infty}\int_0^1 \frac{z^{i_1+\cdots+i_p}t^{i_1+\cdots+i_p-1}\log^{n-1}(t)}{i_1\ldots i_p}dt$$

$$= (-1)^p \sum_{i_1=1}^{\infty}\cdots\sum_{i_p=1}^{\infty} \frac{(-1)^{n-1}(n-1)!z^{i_1+\cdots+i_p}}{i_1\ldots i_p(i_1+\cdots+i_p)^n}$$

Proof of lemma cont'd

By the rearrangement lemma (Hoffman, Pacific J. Math. 1992, Lemma 4.3), this is

$$(-1)^p \sum_{j_1 > j_2 > \cdots > j_p \geq 1} \frac{(-1)^{n-1}(n-1)!p!z^{j_1}}{j_1^{n+1} j_2 \cdots j_p}$$

and the lemma follows. Thus we have

$$S_{n,p}(1) = \zeta(n+1, \{1\}_{p-1})$$

and

$$S_{n,p}(-1) = \zeta(\overline{n+1}, \{1\}_{p-1}).$$

Height one MZVs

Now the "height one" multiple zeta values $\zeta(n, 1, \ldots, 1)$ are well-known. There is the generating function

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\zeta(n+1,\{1\}_{m-1})t^n s^m = \frac{\Gamma(1-t)\Gamma(1-s)}{\Gamma(1-t-s)},$$

and since

$$\frac{\Gamma(1-t)\Gamma(1-s)}{\Gamma(1-s-t)} = 1 - \exp\left(\sum_{i\geq 2}\zeta(i)\frac{t^i+s^i-(t+s)^i}{i}\right)$$

all the $\zeta(n, 1, \ldots, 1)$ are rational polynomials in $\zeta(2), \zeta(3), \ldots$.

Kölbig's identities

Kölbig wrote $s_{n,p}$ and $\sigma_{n,p}$ for $S_{n,p}(1)$ and $(-1)^p S_{n,p}(-1)$ respectively. In (SIAM J. Math. Anal. 1986) he proved that

$$\sum_{j=1}^{n} \binom{n+p-j-1}{p-1} \sigma_{j,n+p-j} + \sum_{j=1}^{p} \binom{n+p-j-1}{n-1} \sigma_{j,n+p-j} = s_{n,p} \quad (1)$$

While this shows that the $s_{n,p}$ can be written in terms of the $\sigma_{n,p}$, the reverse is certainly not true, as witness $\zeta(\bar{3},1) = \sigma_{2,2}$. But certain combinations of the $\sigma_{n,p}$ can be written in terms of the $s_{n,p}$.

I_p in terms of height one MZVs

Our previous theorem can be written as

$$I_p = \sum_{i=1}^{p-1} (-1)^i a_{\lfloor \frac{i}{2} \rfloor} \sigma_{i,p-i}$$

and Kölbig's relations as

$$\sum_{i=1}^{p-1} \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) \sigma_{i,p-i} = s_{j,p-j}.$$

If we can find ρ_j, $1 \le j \le p-1$, so that

$$\sum_{j=1}^{p-1} \rho_j \left(\binom{p-i-j}{p-j-1} + \binom{p-i-1}{j-1} \right) = (-1)^i a_{\lfloor \frac{i}{2} \rfloor} \qquad (2)$$

then I_p can be written in terms of height-one multiple zeta values. Indeed this is the case!

I_p in terms of height one MZVs cont'd

The trick is to note the imposing the symmetry condition $\rho_{p-j} = \rho_j$ reduces the system (2) to

$$\sum_{j=1}^{p-i} \rho_j \binom{p-i-1}{j-1} = \frac{(-1)^i}{2} a_{\lfloor \frac{i}{2} \rfloor}, \ 1 \le i \le p-1. \qquad (3)$$

While the system (3) is dependent, restricting to the last $\lfloor \frac{p}{2} \rfloor$ equations gives a system with unique solution

$$\rho_k = \frac{(-1)^k}{2} \sum_{i=0}^{k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{k-1}{j}, \ 1 \le k \le \lfloor \frac{p}{2} \rfloor.$$

I_p in terms of height one MZVs cont'd

It remains to show that extending this solution for ρ_k to $1 \leq k \leq p-1$ is compatible with the symmetry condition $\rho_k = \rho_{p-k}$, but this turns out to be a consequence of the Euler-polynomial identity

$$(-1)^p \sum_{k=0}^{n} \binom{n}{k} E_{k+p}(0) = (-1)^n \sum_{k=0}^{p} \binom{p}{k} E_{k+n}(0).$$

So we have the following result.

Theorem

For $p \geq 2$, I_p is

$$\frac{(-1)^p}{2} \sum_{k=1}^{p-1} (-1)^k \zeta(k+1, \{1\}_{p-k-1}) \sum_{j=0}^{k-1} \binom{k-1}{j} a_{\lfloor \frac{n-1-j}{2} \rfloor}$$

A conjecture on alternating MZVs

Although the questions about the alternating MZVs coming from the integral are resolved, the cancellation of "exotic" terms suggests that alternating MZVs satisfy the following conjecture.

Conjecture

For positive integers k, m, $\zeta(\overline{2m}, \{1\}_k)$ can be written

$$\zeta(\overline{2m}, \{1\}_k) = P(\zeta(2), \zeta(3), \dots) - \sum_{n=m+1}^{m+\lfloor \frac{k}{2} \rfloor} [x^{2m-2}] E_{2n-3}(x) \zeta(\overline{2n-1}, \{1\}_{k+1-2(n-m)})$$

where P is a rational polynomial function and $[x^k]F(x)$ is the coefficient of x^k in the polynomial $F(x)$.

A conjecture on alternating MZVs cont'd

This conjecture can be seen to hold through weight $2m+k \le 12$ by examining the tables of the Multiple Zeta Value Data Mine (Blümlein, Broadhurst, and Vermaseren 2010). It is also consistent with a number of known results. First, if $k=1$ there are no terms $\zeta(\overline{2n-1}, 1, \dots, 1)$ on the right-hand side, so $\zeta(\overline{2m}, 1)$ should be a rational polynomial in the $\zeta(i)$, and indeed the following result holds.

Theorem (N. Nielsen, 1909)

$$\zeta(\overline{2n}, 1) = \frac{2n-1-(n-1)2^{2n+1}}{2^{2n+1}}\zeta(2n+1) + \sum_{i=1}^{n-1} \frac{2^{2i-1}-1}{2^{2i-1}}\zeta(2i)\zeta(2n-2i+1)$$

Aside: Rewriting Nielsen's identity

Nielsen's identity can be rewritten

$$\zeta(\overline{2n},1)=\frac{\zeta(2n+1)}{2^{2n+1}}+(n-1)\zeta(\overline{2n+1})-\frac{1}{2}\sum_{i=2}^{2n-1}\hat{\zeta}(i)\hat{\zeta}(2n+1-i)$$

where $\hat{\zeta}(i)$ means $\zeta(i)$ if i is odd and $\zeta(\bar{i})$ if i is even. In the same spirit it appears that there are $r_m\in\mathbb{Q}$ such that

$$\zeta(\overline{2m},1,1)=r_m\zeta(2m+2)-\frac{1}{4}\sum_{j=2}^{2m}\hat{\zeta}(j)\hat{\zeta}(2m+2-j)+\frac{1}{6}\sum_{\substack{i+j+k=2m+2\\ i,j,k\geq 2}}\hat{\zeta}(i)\hat{\zeta}(j)\hat{\zeta}(k)+\frac{2m-1}{2}\zeta(\overline{2m+1},1).$$

A conjecture on alternating MZVs cont'd

Second, taking $p = 1$ in Kölbig's identity (1) gives

$$2\zeta(\bar{2}, \{1\}_{n-1}) - \zeta(\bar{3}, \{1\}_{n-2}) + \zeta(\bar{4}, \{1\}_{n-3}) - \cdots \\ + (-1)^{n+1}\zeta(\overline{n+1}) = (-1)^n\zeta(n+1)$$

Assuming the conjecture, the terms that aren't rational polynomials in the $\zeta(i)$ add up to

$$-\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} (E_{2k-1}(0) + E_{2k-1}(1))\zeta(\overline{2k+1}, \{1\}_{n-2k}),$$

which is zero because of the Euler-polynomial identity

$$E_m(x) + E_m(x+1) = 2x^m.$$

A conjecture on alternating MZVs cont'd

Finally, if we assume the conjecture the terms in I_p that aren't polynomial in the $\zeta(i)$ add up to a sum of multiples of $\zeta(\overline{2n+1}, \{1\}_{p-1-2n})$, $2n+1 \le p$, with the coefficient of $\zeta(\overline{2n+1}, \{1\}_{p-1-2n})$ being

$$\frac{(-1)^p}{2}\left[E_{2n+1}(0) - \sum_{j=0}^{n-1} E_{2j+1}(0)[x^{2j}]E_{2n-1}(x)\right].$$

It turns out that

$$E_{2n+1}(0) = \sum_{j=0}^{n-1} E_{2j+1}(0)[x^{2j}]E_{2n-1}(x)$$

is another Euler-polynomial identity.

Remark on generating functions

It is perhaps worth mentioning that the generating function for the alternating MZVs $\zeta(\bar{n}, 1, \ldots, 1)$ is known:

$$\sum_{n,m\geq 1} \zeta(\overline{n+1}, \{1\}_{m-1}) t^n s^m = 1 - {}_2F_1(s, -t; 1-t; -1).$$

This suggests that it might be possible to prove the conjecture by splitting up the generating function somehow. But so far the only thing I've been able to prove is the curiosity

$$\sum_{m=1}^{\infty} \zeta(\overline{n+1}, \{1\}_{m-1}) = \zeta(\bar{n}).$$