The $p$-canonical basis for Hecke algebras and $p$-cells

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Motivation

Notation: \( k = \overline{k} \) field of characteristic \( p \geq 0 \).

Long-standing open problems in modular representation theory (for \( p > 0 \)):

What are the characters of ...

- modular irreducible modules of \( S_r \) over \( k \) for \( p \leq r \)?
- indecomposable tilting modules of \( GL_n \) over \( k \)?

The following basis contains the answer to these questions...
Idea for the $p$-canonical basis

Notation (for $G \supseteq B \supseteq T$ a split, sc alg. group /$k$ with Borel and max. torus):

- the affine Weyl group $W := W_f \rtimes \mathbb{Z}\Phi$ as a Coxeter system $(W, S)$,
- $^k\mathcal{H}$ the Hecke category (defined over $k$ of characteristic $p$),
- $\mathcal{H}$ the Hecke algebra assoc. to $(W, S)$ over $\mathbb{Z}[v, v^{-1}]$.

Theorem (Elias-Williamson, Soergel, Kazhdan-Lusztig, ...)

There exists an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras:

$$\text{ch} : \left[^k\mathcal{H}\right] \longrightarrow \mathcal{H}, \quad [B_s] \rightsquigarrow H_s \text{ for } s \in S$$

where $\left[^k\mathcal{H}\right]$ denotes the split Grothendieck group of $^k\mathcal{H}$.

Definition

The $p$-canonical basis of $\mathcal{H}$ is given by:

$$\{pH_w \mid w \in W\} = \text{ch}(\{\text{self-dual indecomposable objects in } ^k\mathcal{H}\}/\cong).$$
Properties of the $p$-canonical basis

Instead of precisely stating its properties, we give the following slogans:

- The $p$-canonical basis is a positive characteristic analogue of the Kazhdan-Lusztig basis.

- The $p$-canonical basis loses many of the *combinatorial properties* of the KL basis, but preserves its *positivity properties* (as stated in the Kazhdan-Lusztig positivity conjectures).

- The KL-basis (and the KL-polynomials) are ubiquitous in representation theory (e.g. in the *KL-conjectures* relating characters of Verma and simple modules for a semisimple Lie algebra), the $p$-canonical basis is expected to play a similar role in *modular representation theory*. 
The 3-canonical basis in terms of the Kazhdan-Lusztig basis:

\[
\begin{align*}
3H_s &= H_s \\
3H_{st} &= H_{st} \\
3H_{sts} &= H_{sts} \\
3H_{stst} &= H_{st} + H_{stst} \\
3H_{ststs} &= H_s + H_{ststs} \\
3H_{ststst} &= H_{ststs} + H_{stststs} \\
3H_{stststst} &= H_{stst} + H_{stststst}
\end{align*}
\]

Figure: The 3-canonical basis in terms of the Kazhdan-Lusztig basis.

The multiplicities of $\Delta(m)$ in $T(n)$ for $p = 3$.
**p-Cells**

*p*-Cells give a first approximation of the multiplication in the *p*-canonical basis.

**Definition**

Define a pre-order $p \leq_R W$ via:

$$x \leq_R^p y \iff \exists h \in \mathcal{H} : p^H x \text{ occurs with non-zero coefficient in } p^H y h$$

The equivalence classes w.r.t. $\leq$ are called *right* *p*-cells. The left *p*-cell (resp. two-sided) *p*-cell preorder $\leq_L^p$ (resp. $\leq_{LR}^p$) as well as left (resp. two-sided) *p*-cells are defined similarly.
Right $p$-cells in type $\tilde{A}_2$ and $p = 5$
**p-Cells in finite type $A$**

In finite type $A_{n+1}$, we can explicitly describe $p$-cells via the Robinson-Schensted correspondence which establishes a bijection between the symmetric group $S_n$ and pairs of standard tableaux with $n$ boxes mapping $w \in S_n$ to $(P(w), Q(w))$. Following Ariki’s work we can prove:

**Theorem**
*For $x, y \in S_n$ we have:*

\[
\begin{align*}
   x \overset{p}{\sim}_L y & \iff Q(x) = Q(y), \\
   x \overset{p}{\sim}_R y & \iff P(x) = P(y), \\
   x \overset{p}{\sim}_{LR} y & \iff Q(x) \text{ and } Q(y) \text{ have the same shape.}
\end{align*}
\]

*In particular, Kazhdan-Lusztig cells and $p$-cells of $S_n$ coincide.*
References I

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